Exercise 3

A Simple Transformation of curves

1
Solution

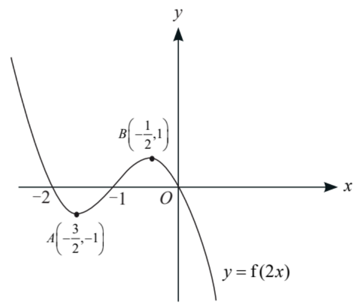
(a) $y = \mathrm{f}(2x)$

Scale the graph parallel to the x-axis by a scale factor of $\frac{1}{2}$.

$A(-3,-1) \to A\left(-\frac{3}{2},-1\right)$

$B(-1,-1) \to B\left(-\frac{1}{2},1\right)$

Remark: multiply x-coordinate by $\frac{1}{2}$.



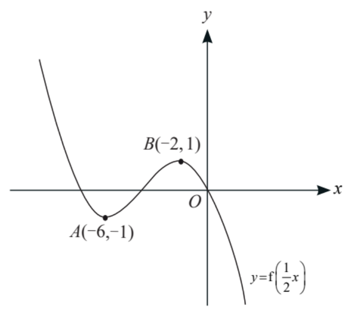
(b) $y = \mathrm{f}\left(\frac{1}{2}x\right)$

Scale the graph parallel to the x-axis by a scale factor of 2.

$A(-3,-1) \to A(-6,-1)$

$B(-1,-1) \to B(-2,1)$

Remark: multiply x-coordinate by 2.



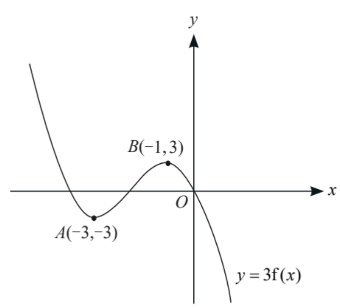
(c) $y = 3\mathrm{f}(x)$

Scale the graph parallel to the y-axis by a scale factor of 3.

$A(-3,-1) \to A(-3,-3)$

$B(-1,-1) \to B(-1,3)$

Remark: multiply y-coordinate by 3.



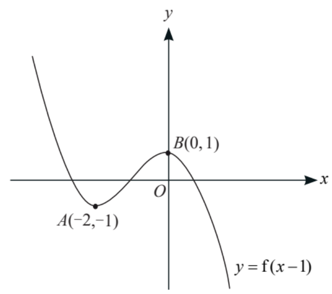
(d) $y = \mathrm{f}(x-1)$

Translates the graph in the positive x direction by 1 unit.

$A(-3,-1) \rightarrow A(-2,-1)$

$B(-1,-1) \rightarrow B(0,-1)$

Remark: add x-coordinate by 1.



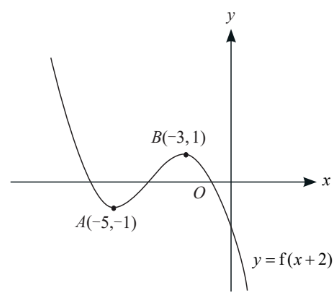
(e) $y = \mathrm{f}(x+2)$

Translates the graph the in negative x direction by 2 units.

$A(-3,-1) \rightarrow A(-5,-1)$

$B(-1,-1) \rightarrow B(-3,-1)$

Remark: add x-coordinate by -2.



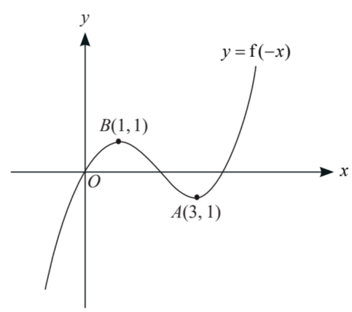
(f) $y = \mathrm{f}(-x)$

Reflects the graph in the y-axis.

$A(-3,-1) \rightarrow A(3,-1)$

$B(-1,-1) \rightarrow B(1,-1)$

Remark: multiply x-coordinate by -1.



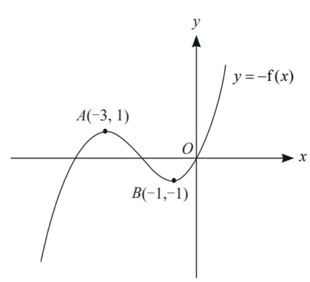
(g) $y = -\mathrm{f}(x)$

Reflects the graph in the x-axis.

$A(-3,-1) \rightarrow A(3,1)$

$B(-1,-1) \rightarrow B(1,-1)$

Remark: multiply x-coordinate by -1.



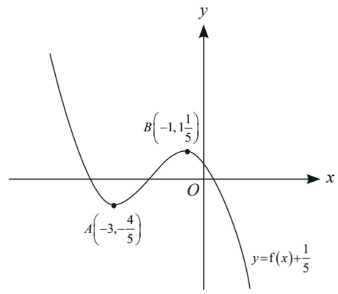
(h) $y=\mathrm{f}(x)+\dfrac{1}{5}$

Translate the graph in the positive y direction by $\dfrac{1}{5}$

$A(-3,-1)\rightarrow A\left(-3,-\dfrac{4}{5}\right)$

$B(-1,-1)\rightarrow B\left(-1,1\dfrac{1}{5}\right)$

Remark: add y-coordinate by $\dfrac{1}{5}$.



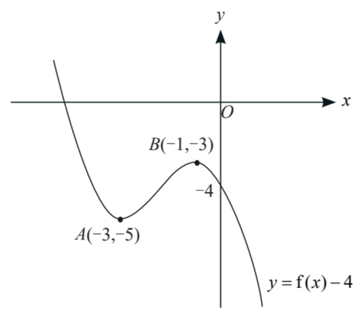
(i) $y=\mathrm{f}(x)-4$

Translate the graph in the negative y direction by 4 units.

$A(-3,-1)\rightarrow A(-3,-5)$

$B(-1,-1)\rightarrow B(-1,-5)$

Remark: add y-coordinate by -4.



2
Solution

(a) $y = 3\mathrm{f}(x+2)$

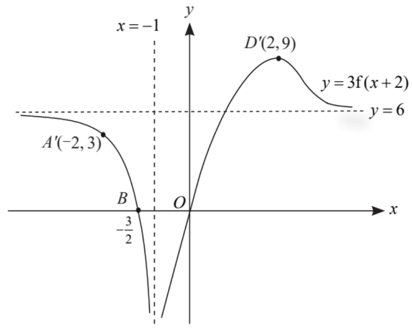
$A(0,1) \to (-2,1) \to (-2,3)$

$B\left(\frac{1}{2},0\right) \to \left(-\frac{3}{2},0\right)$

$C(2,0) \to (0,0)$

$D(4,3) \to (2,3) \to (-2,3) \to (-2,9)$

Asymptotes
$x = 1 \to x = -1 \to x = 1$
$y = 2 \to y = 6$



(b) $y = 3\mathrm{f}(-x+2)$

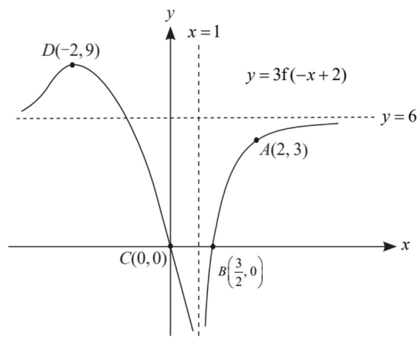
$A(0,1) \to (-2,1) \to (2,1) \to (2,3)$

$B\left(\frac{1}{2},0\right) \to \left(-\frac{3}{2},0\right) \to \left(\frac{3}{2},0\right)$

$C(2,0) \to (0,0)$

$D(4,3) \to (2,3) \to (-2,3) \to (-2,9)$

Asymptotes
$x = 1 \to x = -1 \to x = 1$
$y = 2 \to y = 6$



(c) $y = \mathrm{f}(-x-1)$

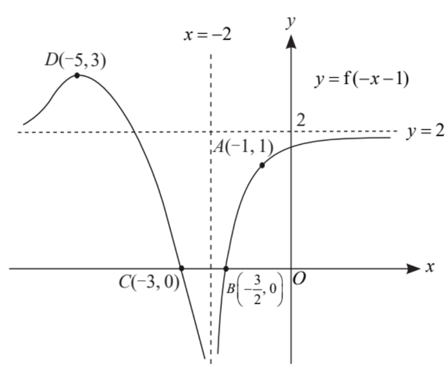
$A(0,1) \to (1,1) \to (-1,1)$

$B\left(\frac{1}{2},0\right) \to \left(\frac{3}{2},0\right) \to \left(-\frac{3}{2},0\right)$

$C(2,0) \to (3,0) \to (-3,0)$

$D(4,3) \to (2,3) \to (-2,3) \to (-2,9)$

Asymptotes
$x = 1 \to x = 2 \to x = -2$



(d) $y=1-4\mathrm{f}(2x)$

$A(0,1)\to(0,1)\to(0,-4)\to(0,-3)$

$B\left(\frac{1}{2},0\right)\to\left(\frac{1}{4},0\right)\to\left(\frac{1}{4},1\right)$

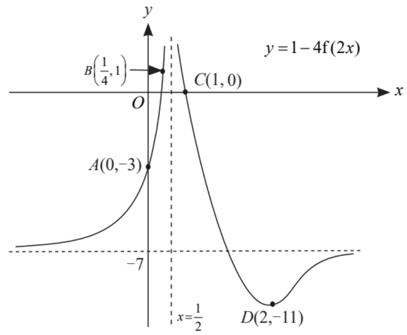
$C(2,0)\to(3,0)\to(-3,0)$

$D(4,3)\to(2,3)\to(2,-12)\to(2,-11)$

Asymptotes

$x=1\to x=\frac{1}{2}$

$y=2\to y=-8\to y=-7$

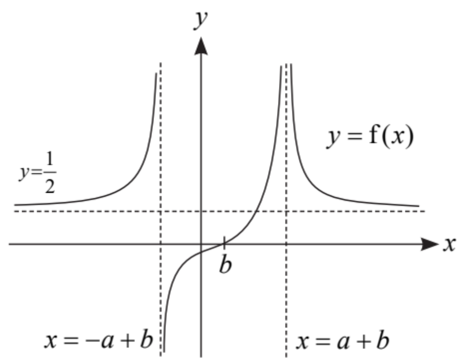


3
Solution

$y = \mathrm{f}(ax+b) \xrightarrow{x \to \frac{x}{a}} y = \mathrm{f}(x+b) \xrightarrow{x \to x-b} y = \mathrm{f}(x)$

Scale parallel to x axis by factor a.

Translate b units in the positive x-direction.



4
Solution

(a) $y = \mathrm{f}(x) \xrightarrow[\text{by factor } \frac{1}{a}]{\text{scaling} // x\text{-axis}} y = \mathrm{f}(ax) \xrightarrow[\text{the positive } x \text{ direction}]{\text{translate } b \text{ units in}} y = \mathrm{f}\big(a(x-b)\big)$

$$A : (2, -3) \to \left(\frac{2}{a}, -3\right) \to \left(\frac{2}{a} + b, -3\right) = (7, -3)$$

$\therefore \quad \frac{2}{a} + b = 7$ .... (1)

$$B : (-3, 1) \to \left(-\frac{3}{a}, 1\right) \to \left(-\frac{3}{a} + b, 1\right) = (-1, 1)$$

$\therefore \quad -\frac{3}{a} + b = -1$ .... (2)

Solving (1) and (2) simultaneously.

$\therefore \quad a = \frac{5}{8}$ and $b = \frac{19}{5}$

(b) $y = \mathrm{g}(x) \to y = \mathrm{g}(x-3) \to y = -2\mathrm{g}(x-3)$

$\therefore \quad \left(\frac{m}{2}, \frac{1}{4}\right) \to \left(\frac{m}{2} + 3, \frac{1}{4}\right) \to \left(\frac{m}{2} + 3, -2\left(\frac{1}{4}\right)\right)$

Comparing the x-coordinate of $\left(\frac{m}{2} + 3, -2\left(\frac{1}{4}\right)\right)$ and $\left(4, \frac{1}{n}\right)$

$$\frac{m}{2} + 3 = 4$$

$$m = 2$$

Comparing the y-coordinate of $\left(\frac{m}{2} + 3, -2\left(\frac{1}{4}\right)\right)$ and $\left(4, \frac{1}{n}\right)$

$$-2\left(\frac{1}{4}\right) = \frac{1}{n}$$

$$n = -2$$

$\therefore \quad m = 2$ and $n = -2$

Alternative Method

Step 1: add 3 to the x - coordinate

Step 2: multiply y-coordinate by -2.

$\therefore \left(\frac{m}{2} + 3, -2\left(\frac{1}{4}\right)\right) = \left(4, \frac{1}{n}\right)$

Comparing the x-coordinate of $\left(\frac{m}{2}+3,-2\left(\frac{1}{4}\right)\right)$ and $\left(4,\frac{1}{n}\right)$

$$\frac{m}{2}+3=4$$
$$m=2$$

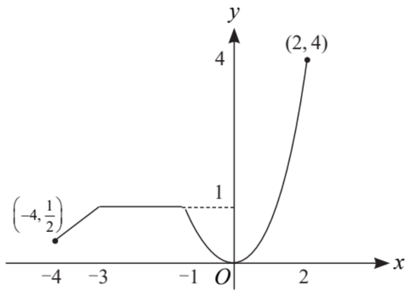
Comparing the y-coordinate of $\left(\frac{m}{2}+3,-2\left(\frac{1}{4}\right)\right)$ and $\left(4,\frac{1}{n}\right)$

$$-2\left(\frac{1}{4}\right)=\frac{1}{n}$$
$$n=-2$$

$\therefore\ m=2$ and $n=-2$

5
Solution

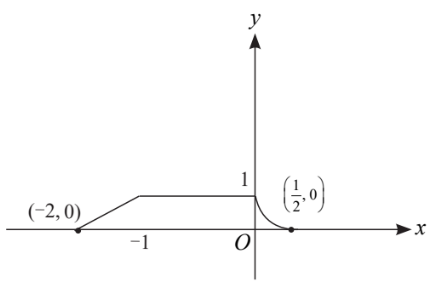
(a) The graph $y = \mathrm{f}(x)$, where $-4 \le x < 2$.



(b) $y = \mathrm{f}(2x - 1)$

Translates the graph in the positive x direction by 1 unit.

Scale the graph parallel to the x-axis by a scale factor of $\dfrac{1}{2}$.



Exercise 3

B Description of Transformations

6
Solution

(a)

Description of sequence of transformations

$$y=\mathrm{f}(x)\xrightarrow{1}y=\mathrm{f}(x+4)\xrightarrow{2}y=\mathrm{f}\left(\frac{1}{2}x+4\right)\xrightarrow{3}y=\mathrm{f}\left(\frac{1}{2}(-x)+4\right)\xrightarrow{4}\frac{y}{3}=\mathrm{f}\left(4-\frac{1}{2}x\right)\longrightarrow y=3\mathrm{f}\left(4-\frac{1}{2}x\right)$$

1. Translate in negative x direction by 4 units
2. Scale parallel to the x-axis by a scale factor of 2
3. Reflect about the y-axis
4. Scale parallel to the y axis by a scale factor of 3

Alternative method

$$y=\mathrm{f}(x)\xrightarrow{1}y=\mathrm{f}(x+4)\xrightarrow{2}y=\mathrm{f}\left(\frac{1}{-2}x+4\right)\xrightarrow{3}\frac{y}{3}=\mathrm{f}\left(4-\frac{1}{2}x\right)\longrightarrow y=3\mathrm{f}\left(4-\frac{1}{2}x\right)$$

1. Translate in negative x direction by 4 units
2. Scale parallel to the x axis by a scale factor of -2
3. Scale parallel to the y axis by a scale factor of 3

Alternative method

$$y=\mathrm{f}(x)\xrightarrow{1}y=\mathrm{f}(-x)\xrightarrow{2}y=\mathrm{f}\left(-(x-4)\right)\xrightarrow{3}y=\mathrm{f}\left(\left(-\frac{1}{2}x\right)+4\right)\xrightarrow{4}\frac{y}{3}=\mathrm{f}\left(4-\frac{1}{2}x\right)\longrightarrow y=3\mathrm{f}\left(4-\frac{1}{2}x\right)$$

1. Reflect about the y - axis
2. Translate in positive x direction by 4 units
3. Scale parallel to the x axis by a scale factor of 2
4. Scale parallel to the y axis by a scale factor of 3

(b)

Description of sequence of transformations

1. Reflection about the y-axis. (accept : Scaling parallel to the x-axis by factor -1)
2. Translate 1 unit in the positive x-direction.
3. Scale parallel to the y-axis by factor 2.

Alternative method

1. Translate 1 unit in the negative x-direction
2. Reflection about the y-axis.
3. Scale parallel to the y-axis by factor 2.

(c)

Description of sequence of transformations

The curve C is translated 4 units in the negative x - direction, followed by 1 stretch parallel to the x - axis with a factor $\frac{1}{2}$, y - axis invariant.

The point $P(a, b)$ is transformed onto $\left(\frac{a-4}{2}, b\right)$.

Alternative method

The curve C is stretch parallel to the x - axis with a factor of $\frac{1}{2}$, y - axis invariant, followed by a translation of 2 units in the negative x - direction.

The point $P(a, b)$ is transformed onto $\left(\frac{a-4}{2}, b\right)$.

7
Solution

(a)

Description of sequence of transformations

$$y=\ln x \xrightarrow{1} y=\ln(x+3) \xrightarrow{2} y=-\ln(x+3) \xrightarrow{3} y-\ln 2=-\ln(x+3) \longrightarrow y=\ln 2-\ln(x+3), \text{ i.e. } y=\ln\left(\frac{2}{x+3}\right).$$

1. Translate in negative x direction by 3 units
2. Reflect about the x - axis
3. Translate in positive y direction by $\ln 2$ units

Alternative method

Description of sequence of transformations

1. Reflect $y=\mathrm{f}(x)$ in the x-axis.
2. Translate the resulting curve 3 units in the negative x-direction.
3. Translate the resulting curve ln2 units in the positive y - direction.

(b) Given $y=\mathrm{e}^{ax^2}-b$

$=\mathrm{e}^{\left(\sqrt{a}x\right)^2}-b$

Let $\mathrm{f}(x)=\mathrm{e}^{x^2}$, then $\mathrm{f}\left(\sqrt{a}x\right)=\mathrm{e}^{\left(\sqrt{a}x\right)^2}$.

i.e. $y=\mathrm{f}(x)\to y=\mathrm{f}\left(\sqrt{a}x\right)\to y=\mathrm{f}\left(\sqrt{a}x\right)+b$

Hence the sequence of transformations are:

1. Scale by a factor of parallel to the x-axis,
2. Translate the resulting curve by b units in the negative y-direction.

8
Solution

$$y = 2\sin(2x+\alpha)\cos(x)$$

$$\downarrow \text{ replace } x \text{ by } x+\alpha$$

$$y = 2\sin(2(x+\alpha)+\alpha)\cos(x)$$
$$y = 2\sin(2x+3\alpha)\cos(x+\alpha)$$

$$\downarrow \text{ replace } x \text{ by } 2x$$

$$y = 2\sin(2(2x)+3\alpha)\cos(2x+\alpha)$$
$$y = 2\sin(4x+3\alpha)\cos(2x+\alpha)$$

$$\downarrow \text{ replace } y \text{ by } -y$$

$$-y = 2\sin(4x+3\alpha)\cos(2x+\alpha)$$
$$y = -2\sin(4x+3\alpha)\cos(2x+\alpha)$$

Description of sequence of transformations

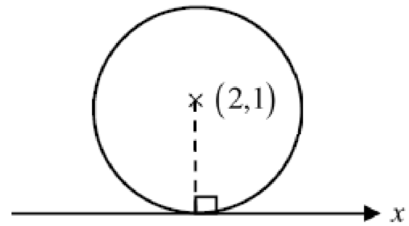
1. Translation of the graph by α units in the negative x-direction, followed by
2. Scaling parallel to x-axis by a scale factor of $\frac{1}{2}$, followed by
3. Reflection in the x-axis.

9
Solution

(a) $\dfrac{(3y-3)^2}{a^2}+\dfrac{(4-2x)^2}{b^2}=1$

$\dfrac{3^2(y-1)^2}{a^2}+\dfrac{(-2)^2(x-2)^2}{b^2}=1$

$\dfrac{(y-1)^2}{\left(\frac{a}{3}\right)^2}+\dfrac{(x-2)^2}{\left(\frac{b}{2}\right)^2}=1$



C is a circle with centre at $(2, 1)$.

Since x - axis is a tangent to circle, radius $= 1$

Hence $a = 3, b = 2$.

Learning point:

Recall: Equation of circle in standard form

$(y-h)^2+(x-k)^2=r^2$

$\dfrac{(y-h)^2}{r^2}+\dfrac{(x-k)^2}{r^2}=1,$ where centre of the circle $=(h,\ k)$ and radius $= r$.

(b) Given $\dfrac{(3y-3)^2}{a^2}+\dfrac{(4-2x)^2}{b^2}=1$

Replace y with $\dfrac{1}{3}y$: $\quad \dfrac{(3y-3)^2}{a^2}+\dfrac{(4-2x)^2}{b^2}=1$

Replace x with $\left(x-\dfrac{1}{2}\right)$: $\quad \dfrac{(y-3)^2}{a^2}+\dfrac{\left(4-2\left(x-\frac{1}{2}\right)\right)^2}{b^2}=1$

$\therefore\ \dfrac{(y-3)^2}{a^2}+\dfrac{(5-2x)^2}{b^2}=1$

Description of sequence of transformations

1. Stretch C parallel to y-axis by factor 3, with x - axis invariant, **then**
2. Translate **resultant curve** by $\dfrac{1}{2}$ units in the positive x direction.

10
Solution

$$x = t^2 + 1,\ y = \ln\left(\sqrt[3]{t} + 1\right)$$
$$\downarrow \text{ Replace } x \text{ with } -x$$
$$-x = t^2 + 1,\ y = \ln\left(\sqrt[3]{t} + 1\right)$$
$$x = -t^2 - 1,\ y = \ln\left(\sqrt[3]{t} + 1\right)$$
$$\downarrow \text{ Replace } x \text{ with } x - 1$$
$$x - 1 = -t^2 - 1,\ y = \ln\left(\sqrt[3]{t} + 1\right)$$
$$x = -t^2,\ y = \ln\left(\sqrt[3]{t} + 1\right)$$
$$\downarrow \text{ Replace } y \text{ with } 2y$$
$$x = -t^2,\ 2y = \ln\left(\sqrt[3]{t} + 1\right)$$
$$x = -t^2,\ y = \frac{1}{2}\ln\left(\sqrt[3]{t} + 1\right)$$
$$x = -t^2,\ y = \ln\sqrt{\sqrt[3]{t} + 1}$$

Description of sequence of transformations

1. Reflect about the y-axis.
2. Translate 1 unit in the positive x-direction.
3. Scale paralle to the y-axis by factor $\frac{1}{2}$.

Alternative method

$$x = t^2 + 1,\ y = \ln\left(\sqrt[3]{t} + 1\right)$$
$$\downarrow \text{ Replace } x \text{ with } x + 1$$
$$x = t^2,\ y = \ln\left(\sqrt[3]{t} + 1\right)$$
$$\downarrow \text{ Replace } x \text{ with } -x$$
$$x = -t^2,\ y = \ln\left(\sqrt[3]{t} + 1\right)$$
$$\downarrow \text{ Replace } y \text{ with } 2y$$
$$x = -t^2,\ y = \frac{1}{2}\ln\sqrt{\sqrt[3]{t} + 1}$$

Description of sequence of transformations

1. Translate 1 unit in the negative x-direction.
2. Reflect the graph about the y-axis.
3. Scale parallel to the y-axis by factor $\frac{1}{2}$.

11
Solution

(a) $y = \dfrac{4x+9}{x+2}$

By performing long division,

$y = 4 + \dfrac{1}{x+2}$, where $a = 4, b = 1$.

Equations of asymptotes are $y = 4$ and $x = -2$.

(b) Given $y = 4 + \dfrac{1}{x+2}$

Replace y with $y+4$: $y = \dfrac{1}{x+2}$

Replace x with $x-2$: $y = \dfrac{1}{x}$

Description of sequence of transformations

Translate the graph 4 units in the negative y - direction,
followed by a translation of 2 units in the positive x - direction.

12
Solution

(a)

Given $\quad y = \ln x$

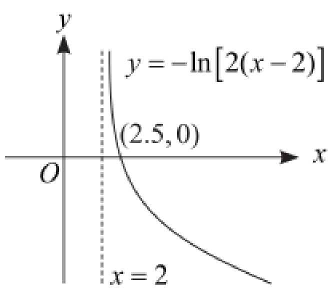
After A: $\quad y = \ln 2x$

After B: $\quad y = \ln\left[2(x-2)\right]$

After C: $\quad y = -\ln\left[2(x-2)\right] = \mathrm{g}(x)$

$\therefore$ the new equation is $y = -\ln\left[2(x-2)\right]$.

(b) The graph of $y = -\ln\left[2(x-2)\right]$



13
Solution

$$
\begin{aligned}
y &= 2x^2 + 4x - 1 \\
&= 2(x+1)^2 - 3 \\
&\quad \downarrow \text{C}': \quad \text{Replace } x \text{ by } x-1 \\
y &= 2x^2 - 3 \\
&\quad \downarrow \text{B}': \quad \text{Replace } y \text{ by } y+3 \\
y &= 2x^2 - 6 \\
&\quad \downarrow \text{A}': \quad \text{Replace } y \text{ by } 2y \\
y &= x^2 - 3
\end{aligned}
$$

$\therefore$ the equation of the original curve is $y = x^2 - 3$.

Alternative method

$$
\begin{aligned}
y &= 2x^2 + 4x - 1 \\
&\quad \downarrow \text{C}': \text{Replace } x \text{ by } x-1 \\
y &= 2(x-1)^2 + 4(x-1) - 1 \\
&= 2(x-1)^2 + 4x - 5 \\
&\quad \downarrow \text{B}': \text{Replace } y \text{ by } y+3 \\
y &= 2(x-1)^2 + 4x - 8 \\
&\quad \downarrow \text{A}': \text{Replace } y \text{ by } 2y \\
y &= (x-1)^2 + 2x - 4 \\
&= x^2 - 2x + 1 + 2x - 4 \\
&= x^2 - 3
\end{aligned}
$$

$\therefore$ the equation of the original curve is $y = x^2 - 3$.

Alternative method

$$
\begin{aligned}
2\mathrm{f}(x+1) + 3 &= 2x^2 + 4x - 1 \\
2\mathrm{f}(x+1) &= 2x^2 + 4x - 4 \\
\mathrm{f}(x+1) &= x^2 + 2x - 2 \\
&= (x+1)^2 - 3 \\
\mathrm{f}(x) &= x^2 - 3
\end{aligned}
$$

14
Solution

Given $y=\dfrac{ax+b}{cx-2}$.

Substitute $(1, 5)$ and $(-8, 0.5)$ into $y=\dfrac{ax+b}{cx-2}$.

At $(1, 5)$: $\quad a+b-5c=-10$ .... (1)
At $(-8, 0.5)$: $8a-b-4c=1$ .... (2)

After transformation, the translated curve is $y=\dfrac{a(x-1)+b}{x(x-1)-2}$. $\quad\triangleleft$ replace x by $x-1$

Substitute $(0,-0.2)$ into $y=\dfrac{a(x-1)+b}{x(x-1)-2}$.

$5a-5b+c=-2$ .... (3)

Solving equations (1), (2) and (3) using GC,

$\therefore\quad a=2, b=3,$ and $c=3$.

Alternative Method

Substitute $(1, 5)$ and $(-8, 0.5)$ into $y=\dfrac{ax+b}{cx-2}$.

At $(1, 5)$: $\quad a+b-5c=-10$ .... (1)
At $(-8, 0.5)$: $8a-b-4c=1$ .... (2)

Since $(0,-0.2)$ lies on the translated curve, then $(-1,-0.2)$ should lie on the original curve.

Substitute $(-1,-0.2)$ into $y=\dfrac{ax+b}{cx-2}$

$5a-5b+c=-2$ .... (3)

Solving equations (1), (2) and (3) using GC,

$\therefore\; a=2, b=3,$ and $c=3$.

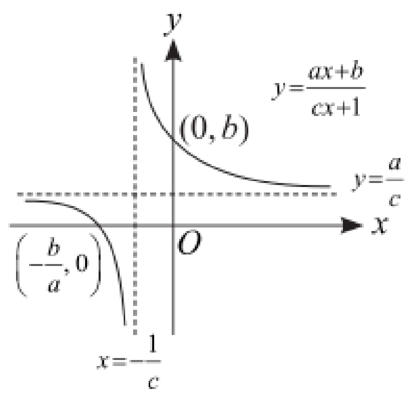
15
Solution

(a) Given $y=\dfrac{ax+b}{cx+1}$

$$y=\frac{a}{c}+\frac{b-\frac{a}{c}}{cx+1}$$

Equations of asymptotes: $x=-\dfrac{1}{c}$ is the vertical asymptote and $y=\dfrac{a}{c}$ is the horizontal asymptote.

Axial intercept : When $y=0,\ x=-\dfrac{b}{a}$ and when $x=0,\ y=b.$



(b) Given $y=\dfrac{ax+b}{cx+1}$

Replace y with $2y$: $2y=\dfrac{ax+b}{cx+1}$ ◁ scaling parallel to y-axis by factor $\dfrac{1}{2}$

$$y=\frac{1}{2}\left(\frac{ax+b}{cx+1}\right)$$

Replace x with $x-2$: $y=y=\dfrac{1}{2}\left[\dfrac{a(x-2)+b}{c(x-2)+1}\right]$ ◁ translation of 2 units in the positive x-direction

The equation of the new curve $y=\dfrac{1}{2}\left[\dfrac{a(x-2)+b}{c(x-2)+1}\right].$

(c) Since the new curve $y=\mathrm{f}(x)$ passes through the points with coordinates $\left(3,\frac{3}{2}\right)$ and $(6,1)$

Substitute $\left(3,\frac{3}{2}\right)$ and $(6,1)$ into $y=\frac{1}{2}\left[\frac{a(x-2)+b}{c(x-2)+1}\right]$.

At $(6,1)$
$$\frac{3}{2}=\frac{1}{2}\left[\frac{a(3-2)+b}{c(3-2)+1}\right]$$
$$3=\frac{a+b}{c+1}$$
$$a+b=3c+3$$
$$a+b-3c=3 \text{ .... (1)}$$

At $\left(3,\frac{3}{2}\right)$
$$1=\frac{1}{2}\left[\frac{a(6-2)+b}{c(6-2)+1}\right]$$
$$2=\frac{4a+b}{4c+1}$$
$$4a+b=8c+2$$
$$4a+b-8c=2 \text{ .... (2)}$$

Since $y=\frac{3}{4}$ is one of the asymptotes of $y=\mathrm{f}(x)$,

$$\therefore \quad \frac{3}{4}=\frac{1}{2}\left(\frac{a}{c}\right)$$
$$\frac{a}{c}=\frac{3}{2}$$
$$2a-3c=0 \text{ .... (3)}$$

Solving equations (1), (2) and (3) using GC.

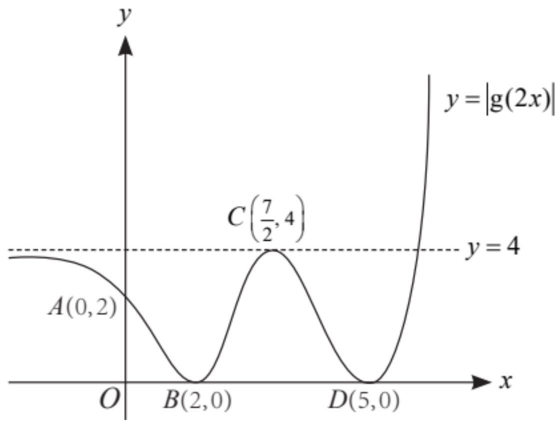
$\therefore\ a=3, b=6$ and $c=2$

Exercise 3

C Sketching Modulus Graphs

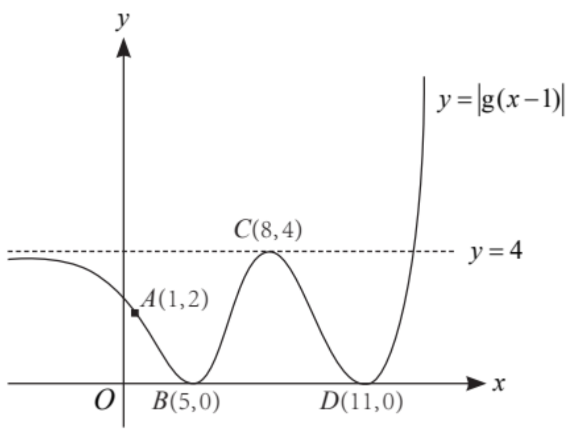
16
Solution

(a) $y = |\,g(2x)\,|$



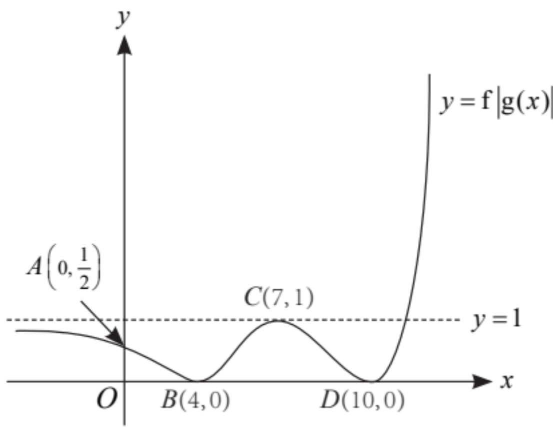
- Divide each y-coordinate by 2.
- Reflect the part of the graph $g(2x)$ which lies below the x-axis in the x-axis.

(b) $y = |\,g(x-1)\,|$



- Add each x-coordinate by 1.
- Reflect the part of the graph $g(x-1)$ which lies below the x-axis in the x-axis.

(c) $y = \frac{1}{4}|g(x)|$



- Multiply each y-coordinate by $\frac{1}{4}$.
- Reflect the part of the graph $g(x)$ which lies below the x-axis in the x-axis.

17
Solution

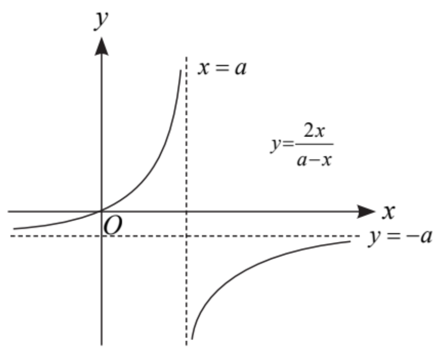
(a)(i) $y = \dfrac{2x}{a-x}$

Equations of asymptotes

$x = a$ is the vertical asymptote

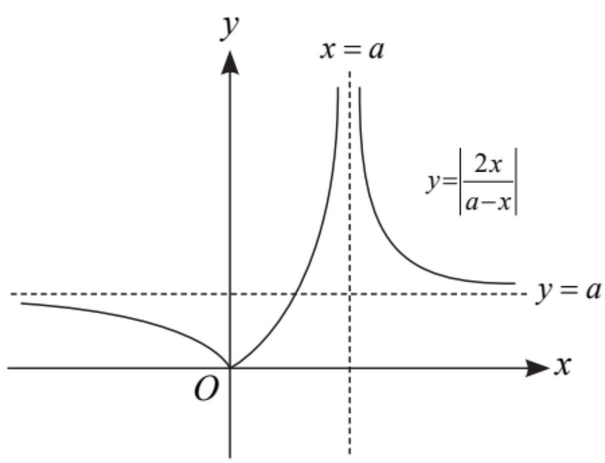
$y = 0$ is the horizontal asymptote.

Axial intercept : When $x = 0$, $y = 0$.

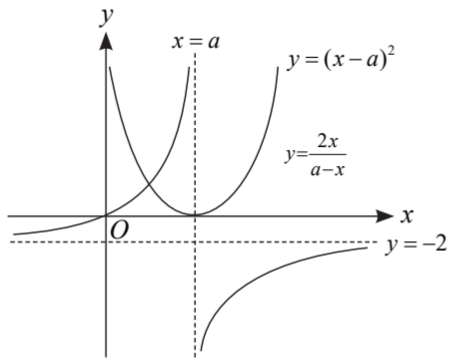


(a)(ii) $y = \left|\dfrac{2x}{a-x}\right|$

- Reflect the part of the graph g(x) which lies below the x-axis in the x-axis.
  (including the horizontal asymptote $y = -a$)



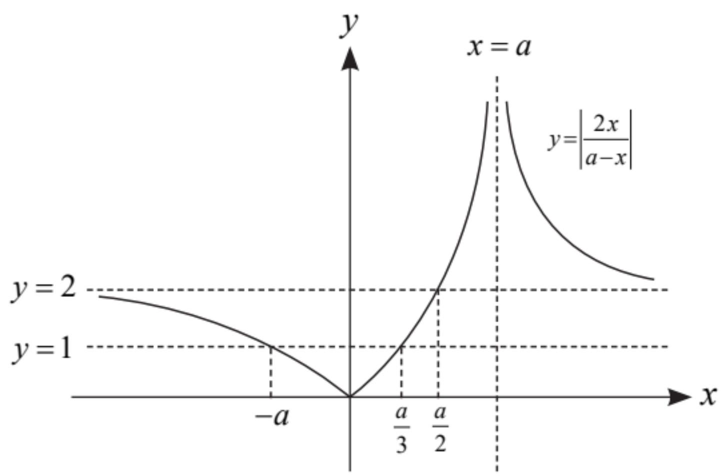
(b) Add the graph $y=(x-a)^2$.



From the above diagram, the curve $y=\dfrac{2x}{a-x}$ and $y=(x-a)^2$ intersect at only one point

Thus the equation $(x-a)^2=\dfrac{2x}{a-x}$ has only one real root.

(c)



Find the point of intersection when $y = 2$ cuts the curve $y = \left|\dfrac{2x}{a-x}\right|$.

Consider $\quad \dfrac{2x}{a-x} = 2$

$$x = \frac{a}{2}$$

Find the point of intersection when $y = 1$ cuts the curve $y = \left|\dfrac{2x}{a-x}\right|$.

Consider $\quad \dfrac{2x}{a-x} = 1 \quad \text{or} \quad \dfrac{2x}{a-x} = -1$

$$x = \frac{a}{3} \quad \text{or} \quad x = -a$$

From the graph, for $1 \le \left|\dfrac{2x}{a-x}\right| \le 2$.

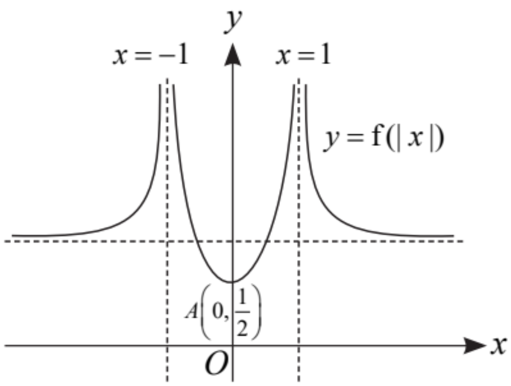
The range of values of x is $x \le -a$ or $\dfrac{a}{3} \le x \le \dfrac{a}{2}$.

Exercise 3

C Sketching Graph of $y = f(|x|)$ from the graph of $y = f(x)$

18
Solution

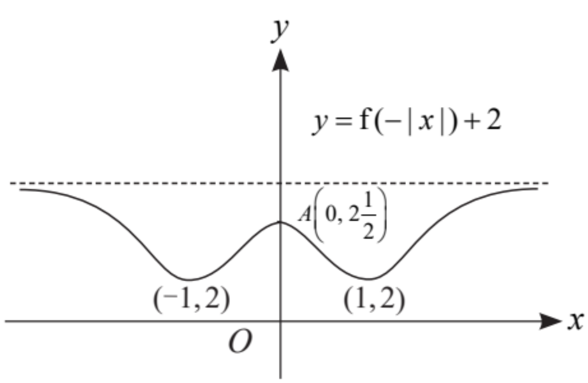
(a) $y = f(|x|)$



Learning point:

- Keep the portion of the graph $y = f(x)$ on and to the right of the y-axis (i.e. for $x \geq 0$) and reflect it about the y-axis to obtain the the graph $y = f(|x|)$.

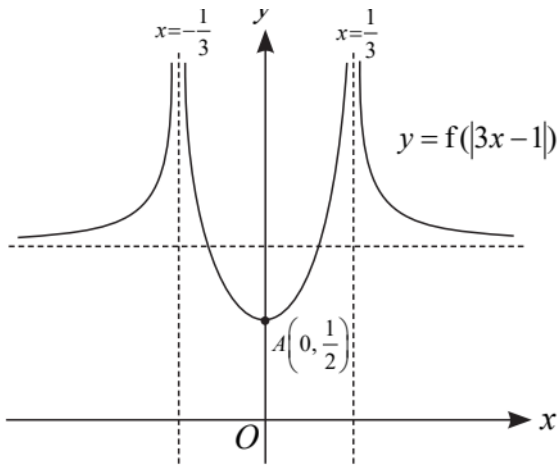
(b) $y = f(-|x|) + 2$



Learning point:

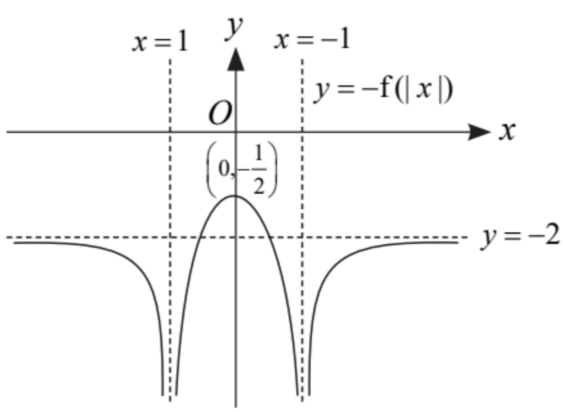
- Reflects the graph $f(x)$ in the y-axis to obtain $y = f(-x)$
- Keep the portion of the graph $y = f(-x)$ on and to the right of the y-axis (i.e. for $x \geq 0$) and reflect it about the y-axis to obtain the the graph $y = f(-|x|)$.
- Translates the graph $y = f(-|x|)$ by 2 units in the positive y-direction to obtain $y = f(-|x|) + 2$.

(c) $y = f(|3x-1|)$



- Keep the portion of the graph $y = f(x)$ on and to the right of the y-axis (i.e. for $x \geq 0$) and reflect it about the y-axis to obtain the the graph $y = f(|x|)$.
- Translates the graph $y = f(|x|)$ by 1 unit in the positive x-direction to obtain $y = f(|x|-1)$.
- Scaling the graph parallel to the x-axis by a factor $\frac{1}{3}$.

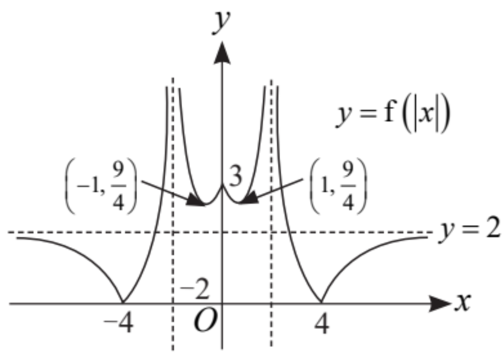
(d) $y = -f(|x|)$



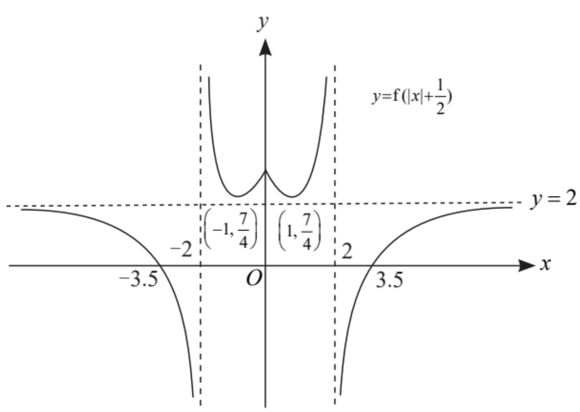
- Keep the portion of the graph $y = f(x)$ on and to the right of the y-axis (i.e. for $x \geq 0$) and reflect it about the y-axis to obtain the the graph $y = f(|x|)$.
- Reflect the resulted graph in the x-axis.

19
Solution

(a) $y = |f(|x|)|$



(b) $y = f(|x| + \frac{1}{2})$

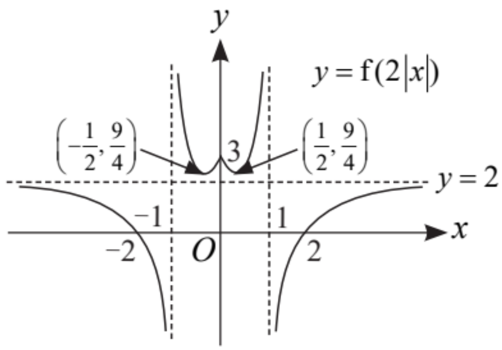


Learning point:

A curve undergoes the following transformations:

$$y = f(x) \longrightarrow y = f\left(x + \frac{1}{2}\right) \longrightarrow y = f\left(|x| + \frac{1}{2}\right)$$

(c) $y = \mathrm{f}(2|x|)$

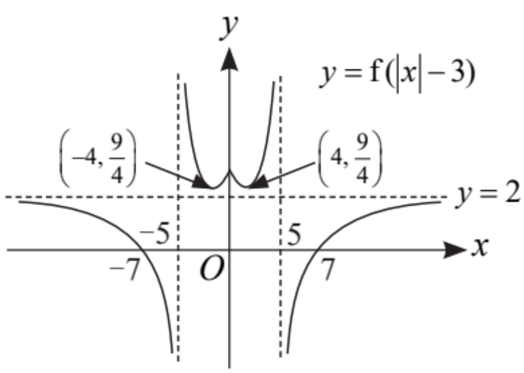


Learning point:

A curve undergoes the following transformations:

$y = \mathrm{f}(x) \longrightarrow y = \mathrm{f}(2x) \longrightarrow y = \mathrm{f}(2|x|)$

(d) $y = \mathrm{f}(|x| - 3)$



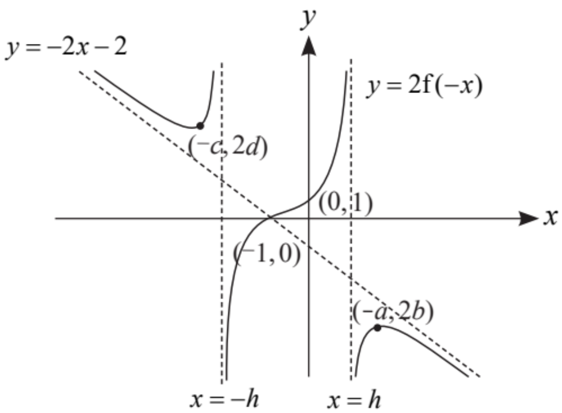
Learning point:

A curve undergoes the following transformations:

$y = \mathrm{f}(x) \longrightarrow y = \mathrm{f}(x-3) \longrightarrow y = \mathrm{f}(|x|-3)$

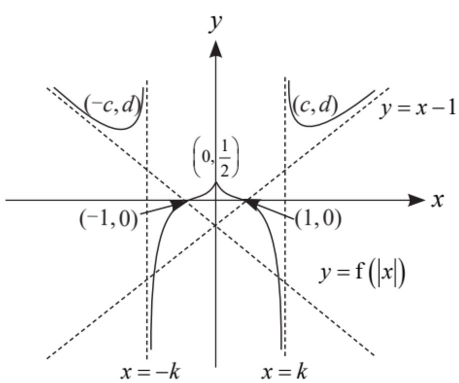
20
Solution

(a) $y = 2\mathrm{f}(-x)$



- Scale parallel to y-axis by factor 2, i.e mutliply all the y-coordinates by 2.
- Reflect in the y-axis.

(b) $y = \mathrm{f}(|x|)$



- Keep the portion of the graph $y = \mathrm{f}(x)$ on and to the right of the y-axis (i.e. for $x \geq 0$) and reflect it about the y-axis to obtain the the graph $y = \mathrm{f}(|x|)$.

21
Solution

(a) $y = \dfrac{2x - x^2}{x^2 - 2x - 3}$

$= -1 - \dfrac{3}{(x-3)(x+1)}$ ◃ perform long division

$\therefore$ equations of the asymptotes of the curve : $y = -1, x = 3, x = -1$

(b) When $x = 0$, $y = 0$

When $y = 0$, $x(2 - x) = 0$

$x = 0$ and $x = 2$

$\therefore$ the coordinates are $(0, 0)$ and $(2, 0)$.

(c) $\dfrac{dy}{dx} = \dfrac{3(2x-2)}{(x^2 - 2x - 3)^2}$

Let $\dfrac{dy}{dx} = 0$

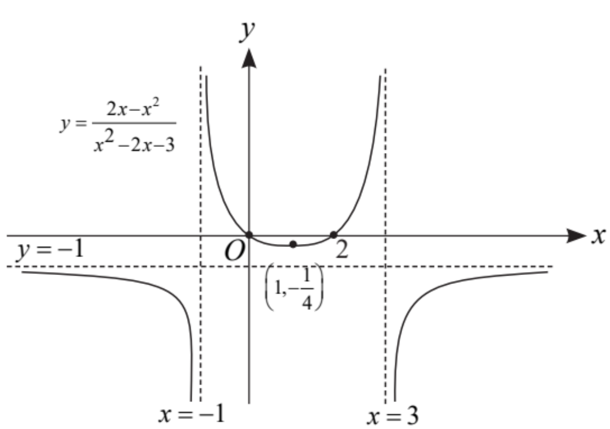
$2x - 2 = 0$

$x = 1$

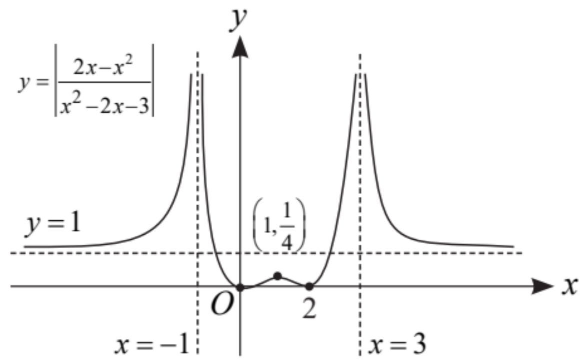
When $x = 1$, $y = -\dfrac{1}{4}$

$\therefore$ the stationary point is $\left(1, -\dfrac{1}{4}\right)$.

The graph of $y = \dfrac{2x - x^2}{x^2 - 2x - 3}$

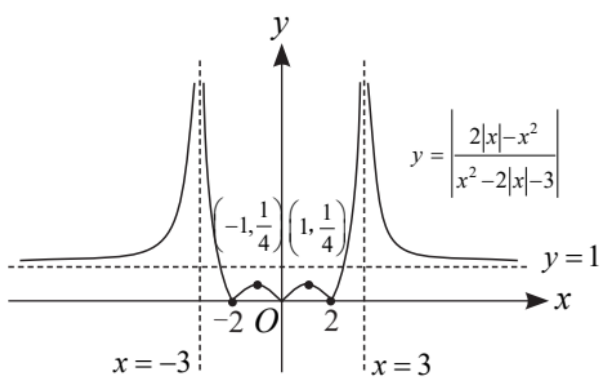


The graph of $y=\left|\dfrac{2x-x^2}{x^2-2x-3}\right|$



- Reflect the part of the graph of $y=\dfrac{2x-x^2}{x^2-2x-3}$ which lies below the x-axis in the x-axis.

The graph of $y=\left|\dfrac{2|x|-x^2}{x^2-2|x|-3}\right|$



- Keep the portion of the graph $y=\mathrm{f}(x)$ on and to the right of the y-axis (i.e. for $x \geq 0$) and reflect it about the y-axis to obtain the the graph $y=\mathrm{f}(|x|)$.

22
Solution

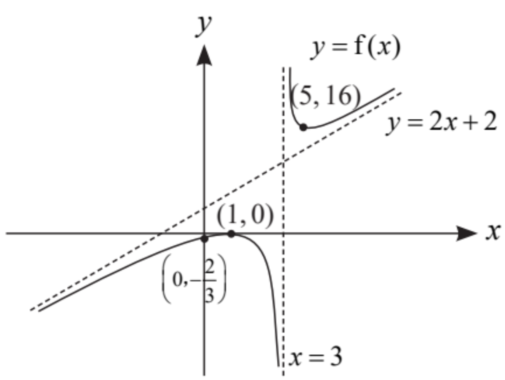
(a) $\mathrm{f}(x)=\dfrac{2(x-1)^2}{x-3}$

$=2x+2+\dfrac{8}{x-3}$

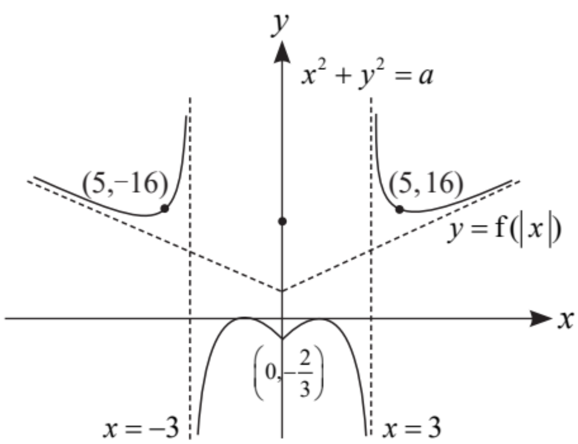
Equations of asymptotes
$x=3$ is the vertical asymptote
$y=2x+2$ is an oblique asymptote.

Axial intercept : When $x=0$, $y=-\dfrac{2}{3}$.
When $y=0$, $x=1$

Determining the turning point
Using GC, (1, 0) maximum point and (5, 16) minimum point.



(b) The graph of $y=\mathrm{f}(|x|)$



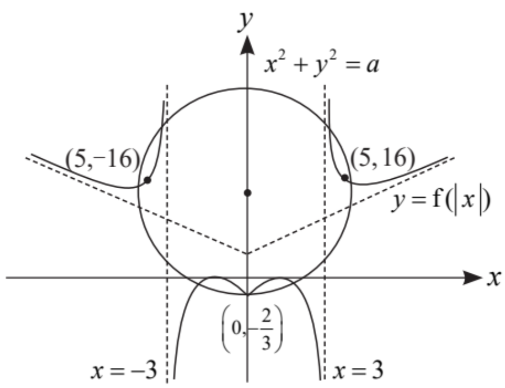
Given $x^2+\left[\dfrac{2(|x|-1)^2}{|x|-3}-k\right]^2=a^2$

Replace $y=\dfrac{2(|x|-1)^2}{|x|-3}$.

$\therefore\ x^2+(y-k)^2=a^2$

Add the the circle $x^2+(y-k)^2=a^2$ to the graph of $y=\mathrm{f}(|x|)$.

Note: Centre of the circle = $(0,\ k)$ and the radius a.

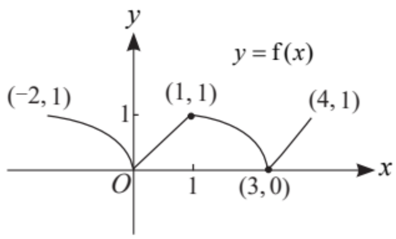


For the equation $x^2+\left[\dfrac{2(|x|-1)^2}{|x|-3}-k\right]^2=a^2$ to have odd number of roots, the radius of the circle needs to be $k+\dfrac{2}{3}$.

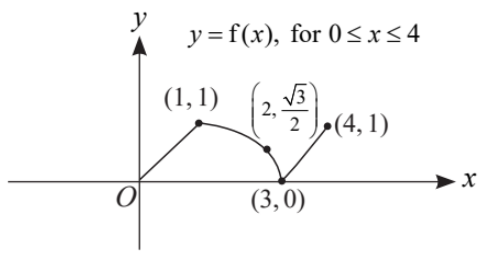
$\therefore\ a=k+\dfrac{2}{3}$

23
Solution

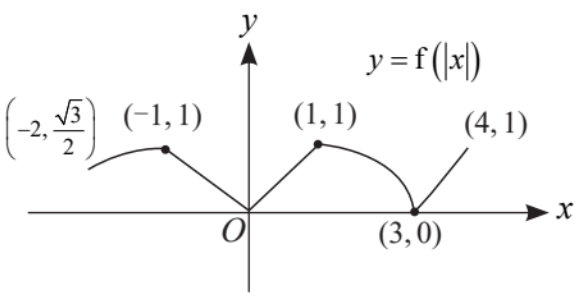
(a) The graph of $y = \mathrm{f}(x)$, where $-2 \le x \le 4$.



(b) First sketch the graph of $y = \mathrm{f}(x)$, for $0 \le x \le 4$. Then reflect the resulted graph to the left of the y-axis in the y-axis to obtain $y = \mathrm{f}\left(|x|\right)$.

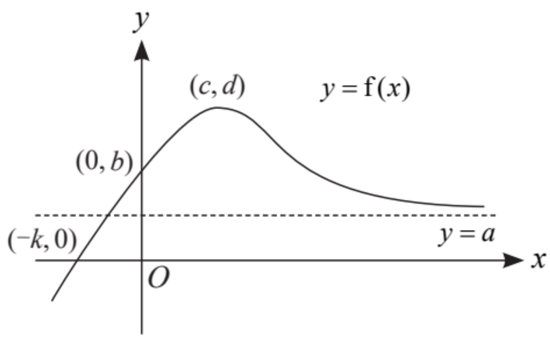


The graph of $y = \mathrm{f}\left(|x|\right)$, where $-2 \le x \le 4$.



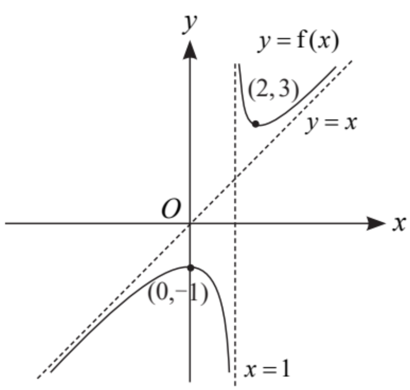
24
Solution

The graph of $y = \mathrm{f}(x)$.



25
Solution

The graph of $y = \mathrm{f}(x)$.

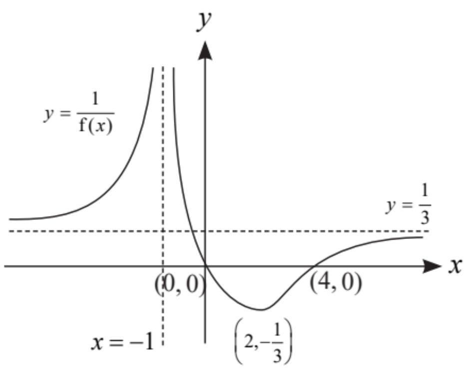


Exercise 3

E Sketch the graph of $y = \dfrac{1}{\mathrm{f}(x)}$ from the graph $y = \mathrm{f}(x)$

26
Solution

The graph of $y = \dfrac{1}{\mathrm{f}(x)}$.



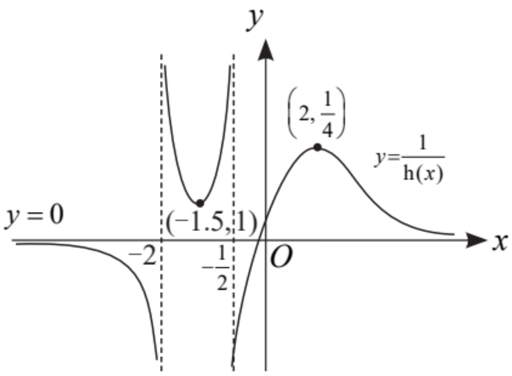
Remark (After transformation)
Coordinates

Maximum point $(2, -3)$ becomes minimum point $\left(2, -\dfrac{1}{3}\right)$.

Asymptotes
$x = 0$ and $x = 4$ becomes x-intercept at 0 and 4 respectively.
$y = 3$ becomes $y = \dfrac{1}{3}$.

27
Solution

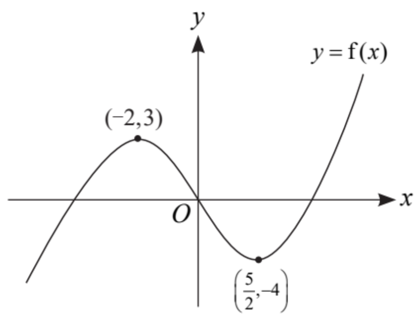
The graph of $y = \dfrac{1}{\mathrm{h}(x)}$.



Remark (After transformation)
Coordinates
Minimum point $(-1.5, 1)$ becomes maximum point $(-1.5, 1)$.
Maximum point $(2, 4)$ becomes minimum point $\left(2, \dfrac{1}{4}\right)$.
Asymptote
The oblique asymptote $y = x + 3$ becomes $y = 0$.

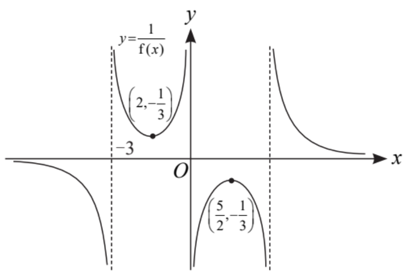
28
Solution

(a) The graph of $y = \mathrm{f}(x)$

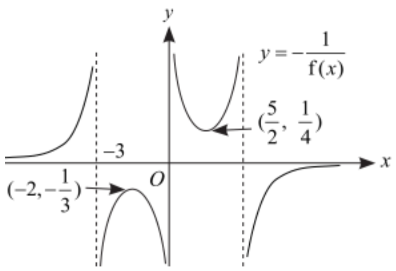


Remark
Translates the graph $y = \mathrm{f}(x+1)$ by 1 unit in the negative x-direction, i.e. minus 1 to all x-coordinates.

(b) Sketch the graph of $y = \dfrac{1}{\mathrm{f}(x)}$ from the graph $y = \mathrm{f}(x)$ follow by reflecting the resulted graph in the x-axis.

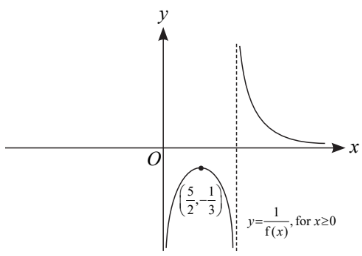


The graph of $y = -\dfrac{1}{\mathrm{f}(x)}$

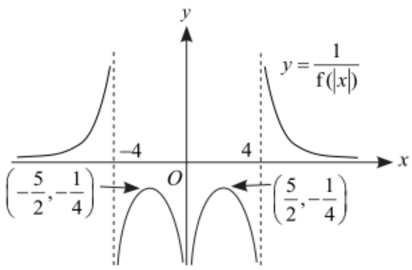


(c) Sketch the graph of $y=\dfrac{1}{\mathrm{f}(x)}$ for $x \ge 0$. (See right)

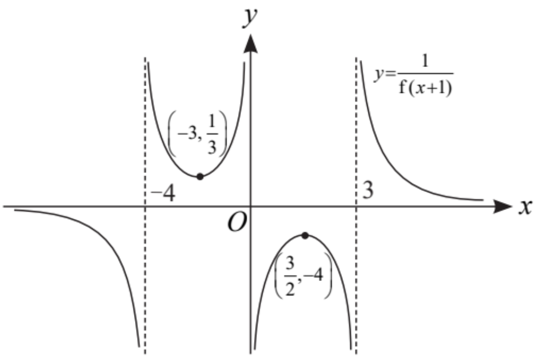
Reflect the resulting graph to the left of the x-axis in the x-axis to obtain the graph of $y=\dfrac{1}{\mathrm{f}(|x|)}$. (See below)



The graph of $y=\dfrac{1}{\mathrm{f}(|x|)}$



(d) The graph of $y=\dfrac{1}{\mathrm{f}(x+1)}$

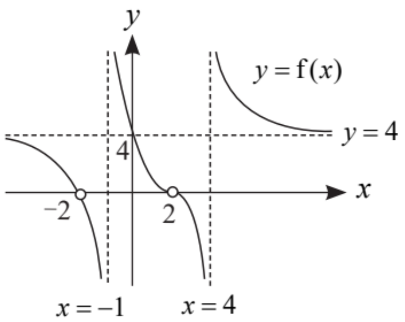


Remark

Translates the graph $y=\dfrac{1}{\mathrm{f}(x+1)}$ by 1 unit in the negative x-direction, i.e. minus 1 to all x-coordinates.

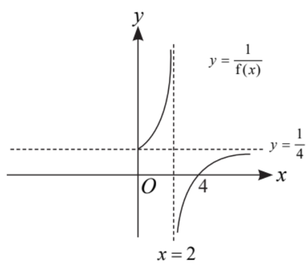
29
Solution

(a) The graph of $y = \mathrm{f}(x)$.



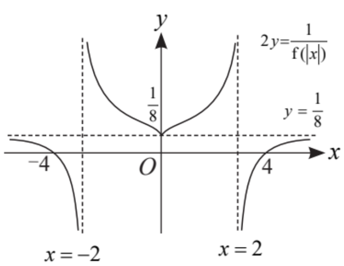
Remark (After transformation)
Coordinates
x-intercepts at -1 and 4 become vertical asymptotes $x = -1$ and $x = 4$ respectively.
Asymptotes
$x = -2$ and $x = 2$ become x-intercepts at -2 and 2 respectively.
$y = 4$ becomes $y = \frac{1}{4}$.

(b) First sketch the graph of $y = \frac{1}{\mathrm{f}(x)}$ for $x \geq 0$. (See right)



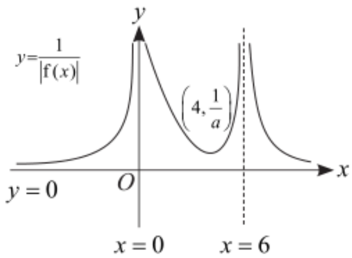
Then graph $y = \frac{1}{\mathrm{f}(|x|)}$ and this follows by graphing $2y = \frac{1}{\mathrm{f}(|x|)}$, i.e. multiplying all the y-coordinates by $\frac{1}{2}$.
(See below)

The graph of $2y = \frac{1}{\mathrm{f}(|x|)}$

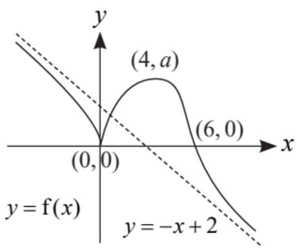


30
Solution

(a) The graph of $y=\dfrac{1}{|\mathrm{f}(x)|}$



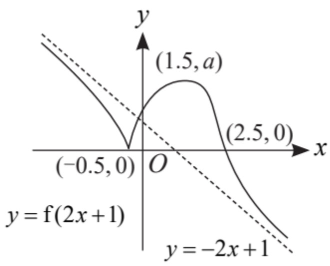
(b) The graph of $y=\mathrm{f}(x)$



Remark

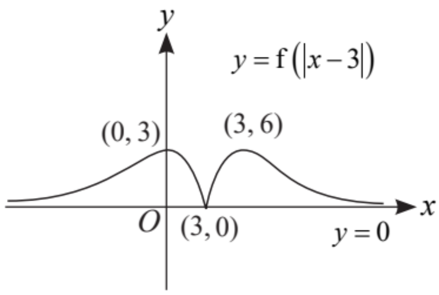
The graph of $y=\mathrm{f}(|x|)$ is the same as $y=\mathrm{f}(x)$ for non-negative x values, so the oblique asymptote of f is $y=-x+2$. Hence the y-values of f are positive for $x<0$.

(c) The graph of $y=\mathrm{f}(2x+1)$



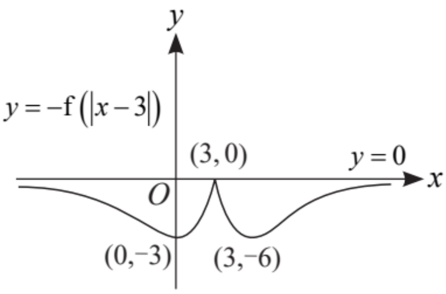
31
Solution

(a) Translates the graph of $y = \mathrm{f}(|x|)$ in the positive x-direction by 3 units to obtain $y = \mathrm{f}(|x-3|)$ as shown on the right.

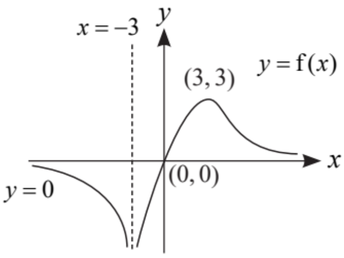


This follows by reflecting the graph $y = \mathrm{f}(|x-3|)$ in the x-axis to obtain the graph $y = -\mathrm{f}(|x-3|)$ as shown below.

The graph $y = -\mathrm{f}(|x-3|)$.



(b) The graph of $y = \mathrm{f}(x)$



32
Solution

(a) Given that $y = \dfrac{1}{\mathrm{g}(x)}$ has $x = 2$ as a vertical asymptote, then $y = \mathrm{g}(x)$ has an x - intercept at $x = 2$.

So $\quad \mathrm{g}(2) = 0$

$$(2)^3 - 3(2) + p = 0$$
$$8 - 6 + p = 0$$

$\therefore\ p = -2$

(b) Given that $y = \dfrac{1}{\mathrm{g}(x)}$ has a minimum point at $y = \dfrac{1}{5}$, then $y = \mathrm{g}(x)$ has a maximum point at $y = 5$.

$$\mathrm{g}'(x) = 3x^2 - 3$$
$$= 3(x+1)(x-1)$$

When $\mathrm{g}'(x) = 0,\ x = \pm 1$.
So turning points of $y = \mathrm{g}(x)$ are at $x = \pm 1$.

By second derivative test,
$\mathrm{g}''(x) = 6x$
When $x = 1,\ \mathrm{g}''(1) = 6(1) > 0$ which is a minimum point.
So the minimum turning point of $\mathrm{g}(x)$ is at $(-1, 5)$.

Substitute $(-1, 5)$ into $\mathrm{g}(x) = x^3 - 3x + p$.

$$(-1)^3 - 3(-1) + p = 5$$
$$-1 + 3 + p = 5$$

$\therefore\ p = 3$

33
Solution

Under the sequence of transformation, the turning point of $y = x^4$ is mapped onto the turning point of $y = \mathrm{f}(x)$.
i.e. the turning point (0, 0) becomes (a, b).

Also, $(0, 0) \xrightarrow[A]{} (l, 0) \xrightarrow[B]{} (l, 0) \xrightarrow[C]{} (l, m)$,
where

A : translation of l units in the positive x - direction
B : stretching by factor k, parallel to the y - axis, x - axis invariant
C : translation of m units in the positive y - direction

By comparing (a, b) and (l, m).
Hence $l = a, m = b.$

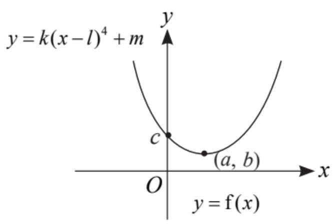
Substitute $x = 0$ and $y = c$ into $y = k(x - l)^4 + m.$

$c = k(0 - a)^4 + b$

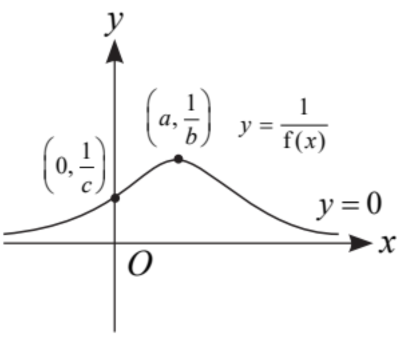
$k = \dfrac{c - b}{a^4}$, since $c > b, k > 0.$

$\therefore\ l = a,\ m = b$ and $k = \dfrac{c - b}{a^4}.$

The graph of $y = \mathrm{f}(x)$



The graph of $y = \dfrac{1}{\mathrm{f}(x)}$



Coordinates of y - intercept $\left(0, \dfrac{1}{c}\right)$

Coordinates of turning point $\left(a, \dfrac{1}{b}\right)$

34
Solution

$y = a(x-1) + \dfrac{b}{x+c}$

$\downarrow$ A (replace y with $-y$)

$-y = a(x-1) + \dfrac{b}{x+c}$

$y = -a(x-1) - \dfrac{b}{x+c}$

$\downarrow$ B (replace x with $x+1$)

$y = -a(x+1-1) - \dfrac{b}{x+1+c}$

$y = -ax - \dfrac{b}{x+1+c}$

$\therefore$ the resulting curve is $y = -ax - \dfrac{b}{x+1+c}$.

Given that y-axis is one of its asymptotes of $y = \mathrm{f}(x)$ has the y-axis,
i.e $x = 0$ is a vertical asymptote.

$1 + c = 0.$

$c = -1$

Therefore, $\mathrm{f}(x) = -ax - \dfrac{b}{x}$. .... (1)

Differentitating $\mathrm{f}(x)$ with respect to x,

$\mathrm{f}'(x) = -a + \dfrac{b}{x^2}$. .... (2)

Given that $\left(1, \dfrac{1}{6}\right)$ is a turning point on $y = \dfrac{1}{\mathrm{f}(x)}$, so, (1, 6) is a turning point on $y = \mathrm{f}(x)$.
Substitute (1, 6) into (1).

$6 = -a(1) - \dfrac{b}{(1)}$

$\therefore \quad -a - b = 6$ .... (3)

When $x = 1, \mathrm{f}'(0) = 0$.
Substitute $x = 1$ and $\mathrm{f}'(0) = 0$ into (2).

$0 = -a(1) - \dfrac{b}{(1)}$

$-a + b = 0$ .... (4)

Solving (3) and (4) using GC.
$\therefore \; a = -3, b = -3$.

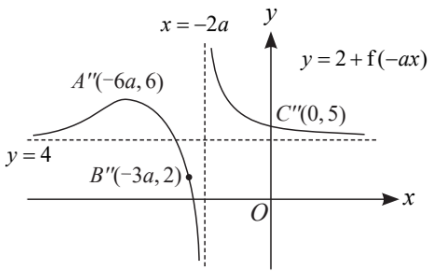
Hence, $a = -3, b = -3$ and $c = -1$.

Exercise 3

F Mixed Exercise

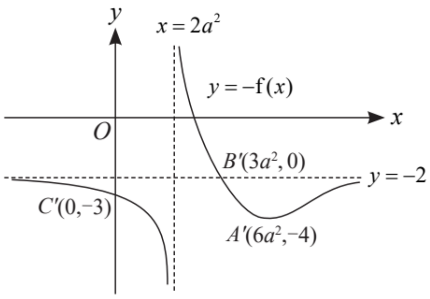
35
Solution

(a)(i) The graph of $y = 2 + \mathrm{f}(-ax)$



Remark
The graph of $y = \mathrm{f}(x+1)$ is obtained from the graph of $y = \mathrm{f}(ax)$ by a translation of 2 units in the positive y-direction i.e add all the y-coordinates by 2.

(ii) The graph of $y = -\mathrm{f}(x)$



Remark
The graph of $y = \mathrm{f}(x+1)$ is obtained from the graph of $y = \mathrm{f}(ax)$ by reflecting in the x-axis.

(b) $y = \mathrm{f}(ax - a^2)$

$= \mathrm{f}\big(a(x-a)\big)$

The graph of $y = \mathrm{f}(ax - a^2)$ can be obtained from the graph of $y = \mathrm{f}(ax)$ by a translation of a units in the positive x-direction.

36
Solution

(a) Translation of 1 unit in the positive y-direction.
Scaling parallel to the x - axis by a factor of 2.

(b) $\tan\theta = \dfrac{x}{2}$ .... (1)

$\sec\theta = \dfrac{y-1}{\sqrt{2}}$ .... (2)

Using Trigogometric Identities, $\tan^2\theta + 1 = \sec^2\theta$ .... (3)
Substituting (1) and (2) into (3).

$$\left(\frac{x}{2}\right)^2 + 1 = \left(\frac{y-1}{\sqrt{2}}\right)^2$$

$\therefore$ the cartesian equation is $\dfrac{(y-1)^2}{\sqrt{2}} - \dfrac{x^2}{4} = 1.$ (Shown)

(c) The cartesian equation is $\dfrac{(y-1)^2}{(\sqrt{2})^2} - \dfrac{x^2}{2^2} = 1.$

Let $\dfrac{(y-1)^2}{(\sqrt{2})^2} - \dfrac{x^2}{2^2} = 0$

$$\frac{y-1}{\sqrt{2}} = \pm\frac{x}{2}$$

$$y = 1 \pm \frac{\sqrt{2}x}{2}$$

$\therefore$ the equations of asymptotes are $y = 1 \pm \dfrac{\sqrt{2}x}{2}$

Alternative method

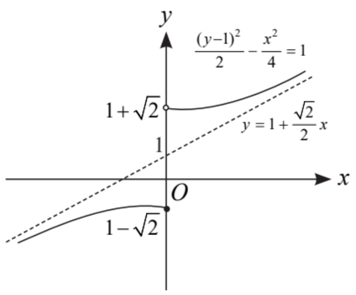
Gradient of asymptotes $= \pm\dfrac{\sqrt{2}}{2}$

Equation of asymptotes : $y = \pm\dfrac{\sqrt{2}x}{2} + c$

Since the asymptotes pass through $(0, 1)$, $\therefore\ c = 1$

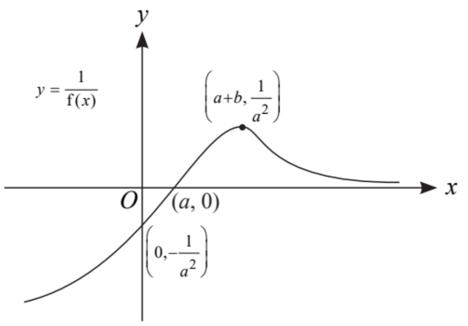
$\therefore$ the equations of asymptotes are $y = 1 \pm \dfrac{\sqrt{2}x}{2}$.

(d)

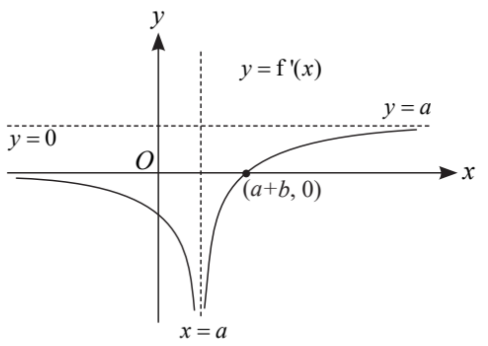


37
Solution

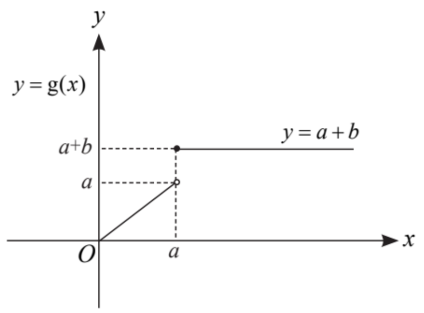
(a) The graph of $y = \dfrac{1}{\mathrm{f}(x)}$



(b) The graph of $y = \mathrm{f}'(x)$



(c) The graph of g



(d) Refer to the graph in **(c)**: Range of $\mathrm{g} = [0, a) \cup \{a + b\}$
Domain of $\mathrm{g} = \mathbb{R} \setminus \{a\}$ (given)
Since $\mathrm{R_g} \subseteq \mathrm{D_f}$, Thus fg exists.

(e) To find range of fg :
$\mathrm{D_{fg}} = \mathrm{D_g} \xrightarrow{\mathrm{g}} \mathrm{R_g} = [0, a) \cup \{a + b\} \xrightarrow{\mathrm{f}} \left(-\infty, -a^2\right] \cup \{a^2\} = \mathrm{R_{fg}}$
$\therefore\ \mathrm{R_{fg}} = \left(-\infty, -a^2\right] \cup \{a^2\}$

38
Solution

(a) $x^2+y^2=1$

$\downarrow$ Replace x by $\dfrac{x}{2}$

$$\left(\frac{x}{2}\right)^2+y^2=1$$

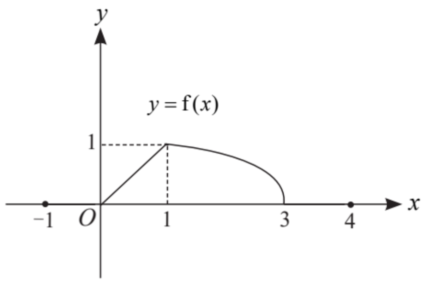
$$\frac{x^2}{4}+y^2=1$$

$\downarrow$ Replace x by $x-1$

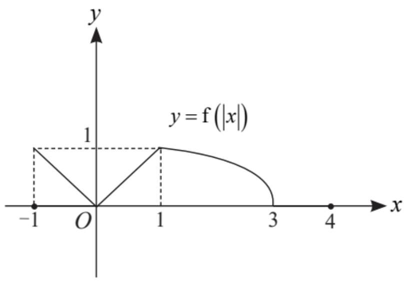
$$\frac{(x-1)^2}{4}+y^2=1$$

Scaling parallel to x - axis by a factor of 2, followed by a translation of 1 unit in the positive x-direction.

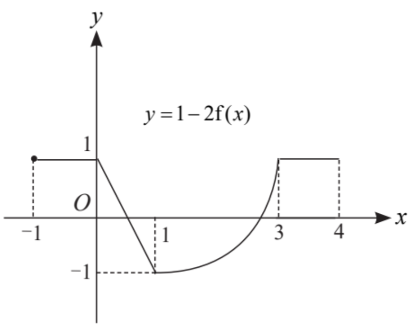
(b)(i) The graph of $y=\mathrm{f}(x)$



(b)(ii) The graph of $y=\mathrm{f}(|x|)$

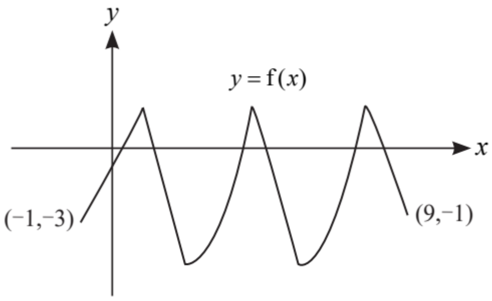


(b)(iii) The graph of $y=1-2\mathrm{f}(x)$



39
Solution

(a) The graph of $y = \mathrm{f}(x)$ for $-1 \le x \le 9$



Learning point:

$\mathrm{f}(9) = \mathrm{f}(5+4) = \mathrm{f}(5) = \mathrm{f}(1+4) = \mathrm{f}(1) = 1 - |3-1| = -1$

$\mathrm{f}(-1) = \mathrm{f}(-1+4) = \mathrm{f}(3) = (3-2)^2 - 4 = -3$

Coordinates of end-points $(-1,-3)$ and $(9,-1)$

From the graph above, the range of $\mathrm{f}(x)$ is $[-4, 1]$.

(b) A : Reflection in the y-axis

$$y = \mathrm{g}(-x) = \frac{1}{1+(-x-1)^2} \quad -2 \le -x \le 2 \qquad \triangleleft \text{ replace } x \text{ by } -x$$

$$= \frac{1}{1+(x+1)^2} \quad -2 \le x \le 2$$

B : Scaling parallel to the x-axis by a factor of 2

$$y = \mathrm{g}\left(-\frac{x}{2}\right) = \frac{1}{1+\left(\frac{x}{2}+1\right)^2} \quad -2 \le \frac{x}{2} \le 2 \qquad \triangleleft \text{ replace } x \text{ by } \frac{x}{2}$$

$$= \frac{4}{4+(x+2)^2} \quad -4 \le x \le 4$$

C : Scaling parallel to the y-axis by a factor of 3

$$\frac{y}{3} = \mathrm{g}\left(-\frac{x}{2}\right) = \frac{12}{4+(x+2)^2} \quad -4 \le x \le 4 \qquad \triangleleft \text{ replace } y \text{ by } \frac{y}{3}$$

$$y = 3\mathrm{g}\left(-\frac{x}{2}\right) = \frac{12}{4+(x+2)^2} \quad -4 \le x \le 4$$

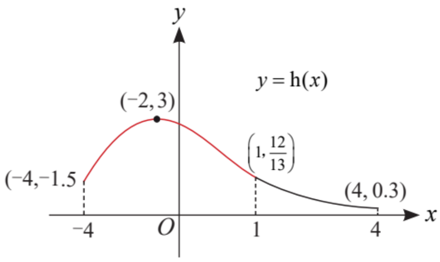
$$\therefore \ \mathrm{h}(x) = \frac{12}{4+(x+2)^2} \text{ and Domain of h} = [-4, 4]$$

(c) Refer to the diagram in **(a)**. Range of $\mathrm{f}(x) = [-4, 1]$

Domain of $\mathrm{h}(x)$ is $[-4, 4]$ (given)

Since $\mathrm{R_f} \subset \mathrm{D_h}$, $\mathrm{hf}(x)$ exists.

First sketch the graph of $y = \mathrm{h}(x)$, for $-4 \le x \le 4$.



Take the $\mathrm{R_f} = [-4, 1]$ as the restricted domain of h and read the corresponding range to obtain $\mathrm{R_{hf}}$.

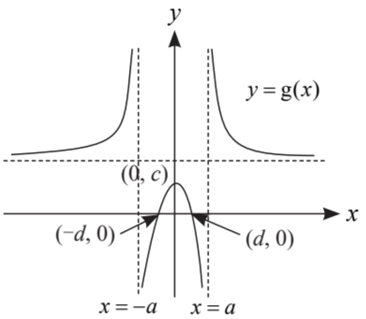
To find range of hf :

$$\mathrm{D_{hf}} = \mathrm{D_f} \xrightarrow{\mathrm{f}} \mathrm{R_f} = [-4, 1] \xrightarrow{\mathrm{h}} \left[\frac{12}{13}, 3\right] = \mathrm{R_{hf}}$$

$\therefore$ Range of $\mathrm{hf}(x) = \left[\frac{12}{13}, 3\right]$

40
Solution

(a) The graph of $y = \mathrm{g}(x)$ for $x \in \mathbb{R}, x \neq \pm a$



Learning point:

- Keep the portion of the graph $y = \mathrm{f}(x)$ on and to the right of the y-axis (i.e. for $x \geq 0$) and reflect it about the y-axis to obtain the the graph $y = \mathrm{g}(x)$.

(b) The line $y = 0$ cuts the graph of $y = \mathrm{g}(x)$ at two points.
Thus g is not a one to one function and hence inverse of g does not exist.

OR

Since $\mathrm{g}(-d) = \mathrm{g}(d) = 0$, g is not a one to one function and hence inverse of g does not exist.

(c) Let $y = \mathrm{h}(x)$, where $x < 0, x \neq -1$

$$= \frac{2|x| - 1}{|x| - 1}$$

Since when $x < 0$, consider $|x| = -x$,

$$\therefore \quad y = \frac{-2x - 1}{-x - 1}$$

$$y = \frac{2x + 1}{x + 1}$$

$$y = 2 - \frac{1}{x + 1}$$

$$\frac{1}{x + 1} = 2 - y$$

$$x = -1 + \frac{1}{2 - y}$$

Thus $\mathrm{h}^{-1}(x) = -1 + \dfrac{1}{2 - x}$

$$\mathrm{h}(x) = \frac{2x - 1}{x - 1}$$

$$= 2 + \frac{1}{x - 1} \quad \text{for } x \in \mathbb{R}, x < 0, x \neq -1.$$

Using GC, $\mathrm{R}_{\mathrm{h}} = (-\infty, 1) \cup (2, \infty)$.

Since $\mathrm{D}_{\mathrm{h}^{-1}} = \mathrm{R}_{\mathrm{h}}$

$\therefore \ \mathrm{D}_{\mathrm{h}^{-1}} = (-\infty, 1) \cup (2, \infty)$

(d) $h^2(x) = hh(x)$

$R_h = (-\infty, 1) \cup (2, \infty)$

$D_h = (-\infty, -1) \cup (-1, 0)$

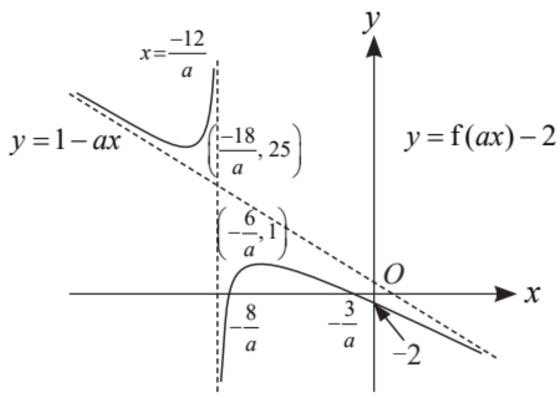
Since $R_h \not\subset D_h$, h^2 does not exist.

Exercise 3

G Higher Order Questions

41
Solution

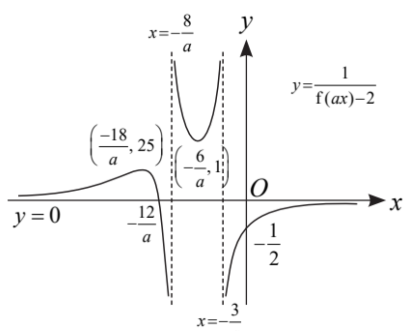
(a) The graph of $y = \mathrm{f}(ax) - 2$



Learning point:
A curve undergoes the following transformations:

- Scale parallel to x-axis by a scale factor of $\frac{1}{a}$, i.e. divide all the x-coordinates by a.
- Translate 2 units in the negative y-direction, i.e. add all the y-coordinates by 2.

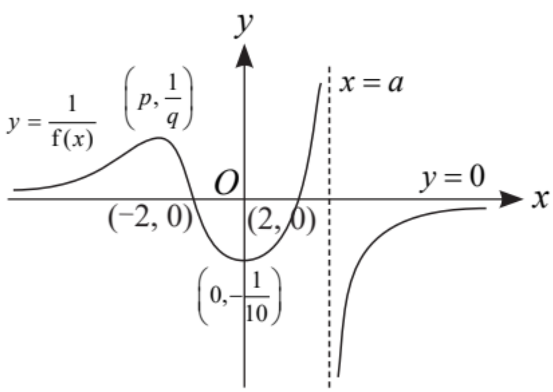
(b) The graph of $y = \dfrac{1}{\mathrm{f}(ax) - 2}$



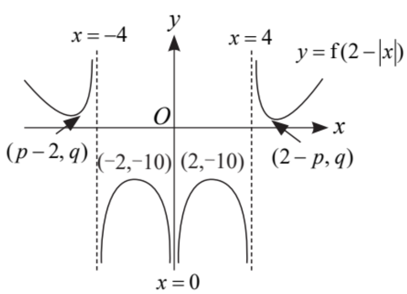
Least value of $k = a$

42
Solution

(a) The graph of $y=\dfrac{1}{\mathrm{f}(x)}$



(b) The graph of $y=\mathrm{f}(2-|x|)$



Learning point:

A curve undergoes the following transformations:

$y=\mathrm{f}(x)\longrightarrow y=\mathrm{f}(x+2)\longrightarrow y=\mathrm{f}(-x+2)\longrightarrow y=\mathrm{f}\left(-|x|+2\right)$

43
Solution

(a) By long division,

$$\mathrm{f}(x)=\frac{x^2+3x+4-a}{x+a}$$

$$=x+(3-a)+\frac{a^2-4a+4}{x+a}$$

The asymptotes are $x=-a$ and $y=x+(3-a)$

For y-intercept, let $x=0$.

i.e. $y=\frac{(0)^2+3(0)+4-a}{(0)+a}$

Then $y=\frac{4-a}{a}$. $\therefore \left(0,\ \frac{4-a}{a}\right)$.

For x-intercept, let $y=0$.

i.e. $\frac{x^2+3x+4-a}{x+a}=0$

Then $x^2+3x+4-a=0$ .... (1)

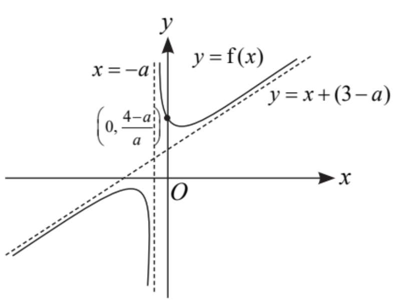
Find the discriminant of (1):

$$3^2-4(4-a)=9-16+4a$$

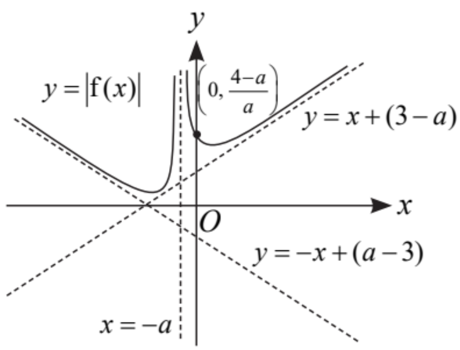
$$=4a-7<0, \text{ where } 0<a<\frac{3}{2}.$$

$\therefore$ There is no solution to the equation thus the graph has no x-intercept.

The graph of $\mathrm{f}(x)=\frac{x^2+3x+4-a}{x+a}$



(b) The graph of $y = |f(x)|$



(c) $f(x) = x + (3-a) + \dfrac{a^2 - 4a + 4}{x+a}$

$$= x + (3-a) + \frac{(a-2)^2}{x+a}$$

$$f'(x) = 1 - \frac{(a-2)^2}{(x+a)^2}$$

At stationary points, $f'(x) = 0$

$$1 - \frac{(a-2)^2}{(x+a)^2} = 0$$

$$(x+a)^2 = (a-2)^2$$

$$x + a = a - 2 \quad \text{or} \quad x + a = -(a-2)$$

$$x = -2 \quad \text{or} \quad x = 2 - 2a$$

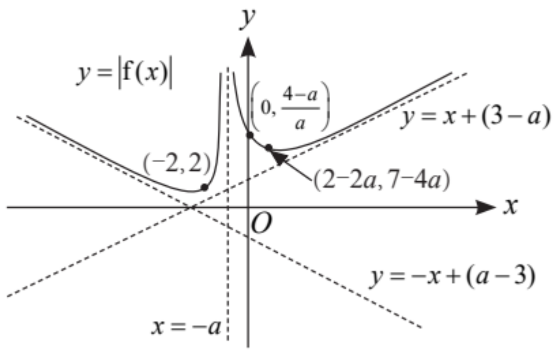
When $x = -2$, $f(-2) = -2 + (3-a) + \dfrac{(a-2)^2}{a-2}$

$$= -1$$

When $x = 2 - 2a$, $f(2-2a) = (2-2a) + (3-a) + \dfrac{(a-2)^2}{(2-2a)+a}$

$$= 7 - 4a$$

On $y=|\mathrm{f}(x)|$, the stationary ponts are $(-2,\ 1)$ and $(2-2a,\ 7-4a)$.



Given in the question that $0<a<\dfrac{3}{2}$.

$\therefore\ 7-4a>1,$ where $0<a<\dfrac{3}{2}$

Hence the line $y=m$ cuts the graph of $y=|\mathrm{f}(x)|$ at two distinct points when $1<m<7-4a$.

Method 2

Consider the equation $\mathrm{f}(x)=m$, where m is a constant.

$$\frac{x^2+3x+4-a}{x+a}=m$$

$$x^2+3x+4-a=mx+am$$

$$x^2+(3-m)x+(4-a-am)=0$$

Using discriminant

$(3-m)^2-4(4-a-m)$

$=m^2-6m+9-16+4a+4am$

$=m^2+(4a-6)m+(4a-7)$

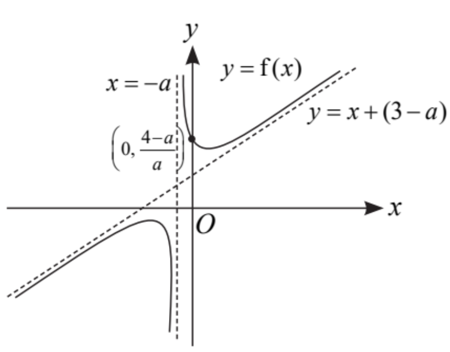
For no real root, Discriminant < 0

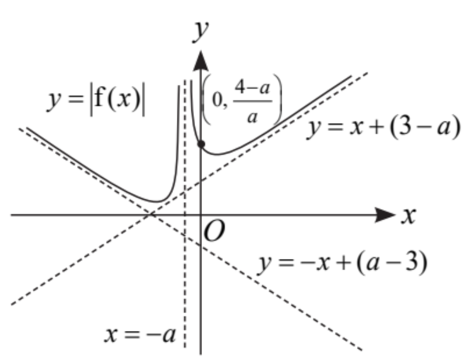
i.e. $\quad (m-(-1))\left[m-(7-4a)\right]<0$

$$-1<m<7-4a$$

Referring to the graph of $y=\mathrm{f}(x)$ and $y=|\mathrm{f}(x)|$.

since $0<a<\dfrac{3}{2},\ \therefore\ 7-4a>1.$

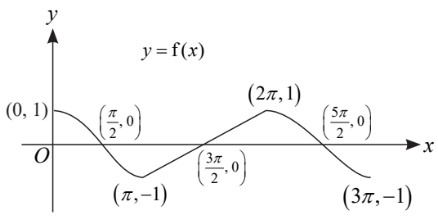




We can deduce that $1<m<7-4a$ for $|\mathrm{f}(x)|=m$ to have exactly two distinct roots.

44
Solution

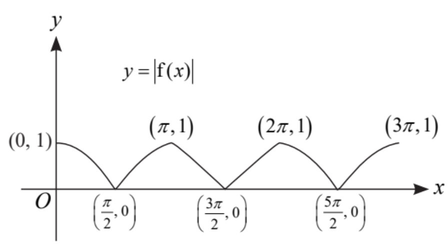
(a) The graph of $y = f(x)$ for $0 \le x \le 3\pi$.



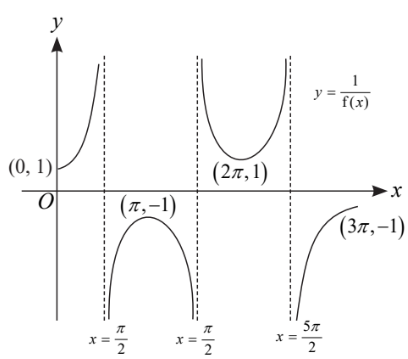
When $-1 < k < 1$ such that $f(x) = k$.
From the graph, we see that there are 3 points of intersection.

Number of solutions = 3

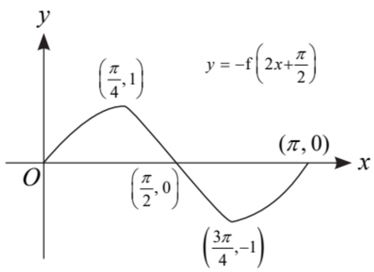
(b)(i) The graph of $y = |f(x)|$ for $0 \le x \le 3\pi$.



(b)(ii) The graph of $y = \dfrac{1}{f(x)}$ for $0 \le x \le 3\pi$.



(b)(iii) The graph of $y = -\mathrm{f}\left(2x + \dfrac{\pi}{2}\right)$ for $0 \le x \le \pi$.



45
Solution

(a) $y = \dfrac{(x+2)^2}{x+1}$

$\downarrow$ (Replace x with $x-2$) (Translation of 2 units in the positive x-direction)

$y = \dfrac{[(x-2)+2]^2}{(x-2)+1}$

$y = \dfrac{x^2}{x-1}$

$\downarrow$ (Replace x with px) (Scaling of scale factor $\dfrac{1}{p}$ along the x-axis)

$y = \dfrac{(px)^2}{(px)-1}$

$y = \dfrac{p^2x^2}{px-1}$

$\downarrow$ (Replace y with $y-q$) (Translation of q units in the positive y-direction)

$y - q = \dfrac{p^2x^2}{px-1} + q$

$y = \dfrac{p^2x^2}{px-1} + q$

After the transformations, the minimum turning point becomes

$$(0, 4) \xrightarrow{\text{(Translation of 2 units in the positive } x\text{-direction)}} (2, 4) \xrightarrow{\text{(Scaling of scale factor } \frac{1}{p} \text{ along the } x\text{-axis)}} \left(\frac{2}{p}, 4\right) \xrightarrow{\text{(Translation of } q \text{ units in the positive } y\text{-direction)}} \left(\frac{2}{p}, 4+q\right).$$

It is given that the point $(0, 4)$ on C_1 after the transformations correponds to (a, b) on C_2.

By comparing $\left(\dfrac{2}{p}, 4+q\right)$ and (a, b).

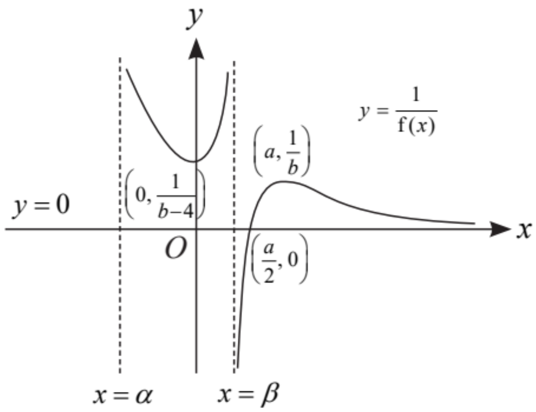
x-coordinate: $\dfrac{2}{p} = a$ $\qquad$ y-coordinate: $\dfrac{2}{p} = a$

$p = \dfrac{2}{a}$ $\qquad$ $4 + q = b$

$q = b - 4$

$\therefore\ p = \dfrac{2}{a}$ and $q = b - 4$

(b) The graph of $y = \dfrac{1}{\mathrm{f}(x)}$



Learning point:

After the transformation:

- Maximum point $(0, q)$ becomes minimum point $\left(0, \dfrac{1}{b-4}\right)$
- Minimum point $(0, q)$ becomes maximum point $\left(a, \dfrac{1}{b}\right)$
- The x-intercepts α and β become vertical asymptotes $x = \dfrac{1}{\alpha}$ and $x = \dfrac{1}{\beta}$ respectively.
- The vertical asymptote $x = \dfrac{1}{p}$ becomes x-intercept.

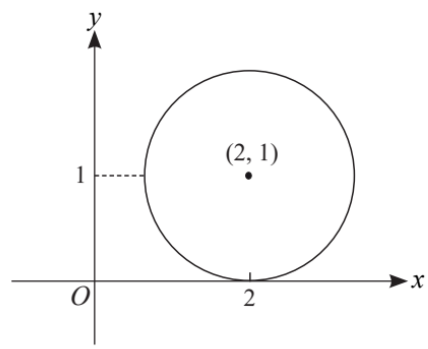
46
Solution

(a) Given $\dfrac{(3y-3)^2}{a^2}+\dfrac{(4-2x)^2}{b^2}=1$

$$\frac{3^2(y-1)^2}{a^2}+\frac{(-2)(x-2)^2}{b^2}=1$$

$$\frac{(y-1)^2}{\left(\frac{a}{3}\right)^2}+\frac{(x-2)^2}{\left(\frac{b}{-2}\right)^2}=1$$

From the above equation, the centre of the circle C is at (2, 1).
Given that x-axis is a tangent to circle, therefore the radius $=1$.



Hence $\left(\dfrac{a}{3}\right)^2=1$ and $\left(\dfrac{b}{-2}\right)^2=1$

$\dfrac{a}{3}=\pm 1$ and $\dfrac{b}{-2}=\pm 1$

Since, $a,b>0$

$a=3,$ $b=2$

(b) Given $\dfrac{(3y-3)^2}{a^2}+\dfrac{(4-2x)^2}{b^2}=1$

$\downarrow$ (Replace y with $\frac{1}{3}y$)

$$\frac{(3y-3)^2}{a^2}+\frac{(4-2x)^2}{b^2}=1$$

$\downarrow$ (Replace x with $\left(x-\frac{1}{2}\right)$)

$$\frac{(y-3)^2}{a^2}+\frac{\left(4-2\left(x-\frac{1}{2}\right)\right)^2}{b^2}=1$$

$$\therefore\ \frac{(y-3)^2}{a^2}+\frac{(5-2x)^2}{b^2}=1$$

Description of sequence of transformations

Stretch C parallel to y-axis by factor 3, with x-axis invariant, then translate resultant curve by $\frac{1}{2}$ units in the positive x direction.

47
Solution

(a) Given $x = \dfrac{k}{t},\ y = t^2 + t,\ t \in \mathbb{R},\ k > 0$

As $x \to \pm\infty,\ t \to 0,\ y \to 0.$

$\therefore\ y = 0$ is a horizontal asymptote.

As $x \to 0,\ t \to \pm\infty,\ y \to \infty.$

$\therefore\ x = 0$ is a vertical asymptote.

(b) Given $x = \dfrac{k}{t}$ .... (1)

and $y = t^2 + t$ .... (2)

Differentiating (1) with respect to t

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{k}{t^2}$$

Differentiating (2) with respect to t

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 2t + 1$$

Using Chain Rule,

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}t} \times \frac{\mathrm{d}t}{\mathrm{d}x} \\ &= (2t+1) \times \frac{t^2}{k} \\ &= \frac{1}{k}t^2(2t+1)\end{aligned}$$

$$\therefore\ \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{k}t^2(2t+1)$$

Given that C has exactly one minimum point, let $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0.$

$$\therefore\quad \frac{1}{k}t^2(2t+1) = 0$$

$$2t + 1 = 0 \quad \text{or} \quad t = 0 \quad (\text{Rejected since } t \neq 0)$$

$$t = -\frac{1}{2}$$

Substitute $t=-\dfrac{1}{2}$ into (1) and (2)

From (1): $x=\dfrac{k}{-\dfrac{1}{2}}$

$=-2t$ and

From (2): $y=\left(-\dfrac{1}{2}\right)^2+\left(-\dfrac{1}{2}\right)$

$=-\dfrac{1}{4}$

Coordinates of minimum point at $t=-\dfrac{1}{2}$ is $\left(-2k,\ -\dfrac{1}{4}\right)$. (Shown)

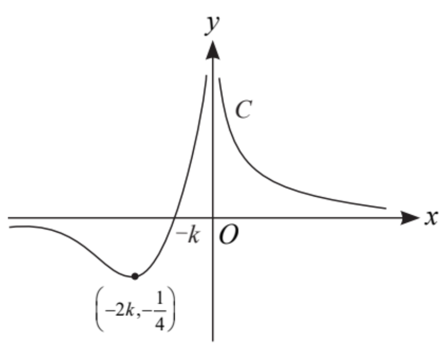
(c) Determine the x intercept by letting $y=0$.

$t(t+1)=0$

$t=-1$ ($\because$ $t\neq 0$)

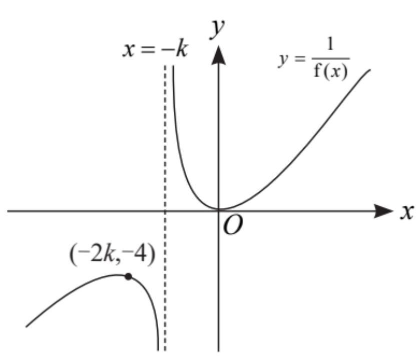
Substitute $t=-1$ into (1).

$\therefore$ $x=-k$



(d) Deduce from graph in part **(c)**.

The graph of $y=\dfrac{1}{\mathrm{f}(x)}$.



48
Solution

$$y = a(x-1) + \frac{b}{x+c}$$

$\downarrow$ A (replace y with $-y$)

$$-y = a(x-1) + \frac{b}{x+c}$$

$$y = -a(x-1) - \frac{b}{x+c}$$

$\downarrow$ B (replace x with $x+1$)

$$y = -a(x+1-1) - \frac{b}{x+1+c}$$

$$y = -ax - \frac{b}{x+1+c}$$

$\therefore$ the resulting curve is $f(x) = -ax - \dfrac{b}{x+1+c}$. .... (1)

To obtain the vertical asymptote, let the denominator of $f(x) = -ax - \dfrac{b}{x+1+c}$ be zero.

i.e. $x+1+c = 0$

$x = -(1+c)$

Given that y-axis as one of its asymptotes, i.e. $x = 0$ is a vertical asymptote.
Comparing $-(1+c)$ with 0
So $-(1+c) = 0$

$c = -1$

Substitute $c = -1$ into (1).

$\therefore$ $f(x) = -ax - \dfrac{b}{x}$

Differentiate (1) with respect to x.

$f'(x) = -a + \dfrac{b}{x^2}$ .... (2)

Given that $\left(1, \dfrac{1}{6}\right)$ is a turning point on $y = \dfrac{1}{f(x)}$, then coordinates (1, 6) is the turning point on $y = f(x)$.
Substitute (1, 6) into (2).
$\therefore$ $-a - b = 6$ .... (3)

When $x = 1$, $f'(0) = 0$.
Substitute $x = 1$ and $f'(0) = 0$ into (2).
$-a + b = 0$. .... (4)

Solving (3) and (4) using GC: $a = -3$ and $b = -3$.

$\therefore$ $a = -3$, $b = -3$ and $c = -1$.

49
Solution

(a) $y = \dfrac{ax^2 - bx}{x^2 - c}$ .... (1)

$y = a + \dfrac{ac - bx}{x^2 - c}$

Since $y = 2$ is a horizontal asymptote, $a = 2$.
Since $x = -2$ is a vertical asymptote, $c = 4$.

Substitute $a = 2$ and $c = 4$ into (1).

$\therefore\ y = \dfrac{2x^2 - bx}{x^2 - 4}$

Given that the curve passing through $\left(3, \dfrac{9}{5}\right)$, substituting this point into $y = \dfrac{2x^2 - bx}{x^2 - 4}$.

$$\therefore\ \frac{9}{5} = \frac{2(3)^2 - b(3)}{(3)^2 - 4}$$
$$b = 3$$

$\therefore\ a = 2,\ b = 3$ and $c = 4$.

(b) $y = \dfrac{2x^2 - 3x}{x^2 - 4}$

$y(x^2 - 4) = 2x^2 - 3x$

$(y - 2)x^2 + 3x - 4y = 0$ .... (1)

For the values of y that C_1 cannot take, it implies that there is no real roots in (1).
i.e. Discriminant of (1) < 0

$$(3)^2 - 4(y-2)(-4y) < 0$$
$$16y^2 - 32y + 9 < 0$$

Method 1 (Using quadratic formula)

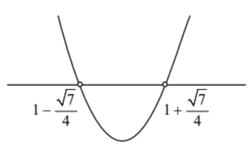
Let $16y^2 - 32y + 9 = 0$

$$\therefore\quad y = \frac{32 \pm \sqrt{(32)^2 - 4(16)(9)}}{2(16)}$$
$$= 1 \pm \frac{\sqrt{7}}{4}$$

$$\left(y - 1 + \frac{\sqrt{7}}{4}\right)\left(y - 1 - \frac{\sqrt{7}}{4}\right) = 0$$

Since $16y^2 - 32y + 9 < 0$

$$\left(y - 1 + \frac{\sqrt{7}}{4}\right)\left(y - 1 - \frac{\sqrt{7}}{4}\right) < 0$$
$$1 - \frac{\sqrt{7}}{4} < y < 1 + \frac{\sqrt{7}}{4}$$



$\therefore$ the set of values of y is $\left\{ y \in \mathbb{R} : 1 - \dfrac{\sqrt{7}}{4} < y < 1 + \dfrac{\sqrt{7}}{4} \right\}$

Method 2 (Completing the square)

$$16y^2 - 32y + 9 < 0$$

$$y^2 - 2y + \frac{9}{16} < 0$$

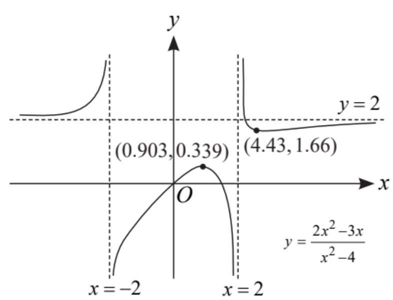
$$(y-1)^2 - \frac{7}{16} < 0$$

$$\left(y - 1 + \frac{\sqrt{7}}{4}\right)\left(y - 1 - \frac{\sqrt{7}}{4}\right) < 0$$

$\therefore\ 1 - \frac{\sqrt{7}}{4} < y < 1 + \frac{\sqrt{7}}{4}$

$\therefore$ the set of values of y is $\left\{ y \in \mathbb{R} : 1 - \frac{\sqrt{7}}{4} < y < 1 + \frac{\sqrt{7}}{4} \right\}$

(c) The graph of $y = \frac{2x^2 - 3x}{x^2 - 4}$.

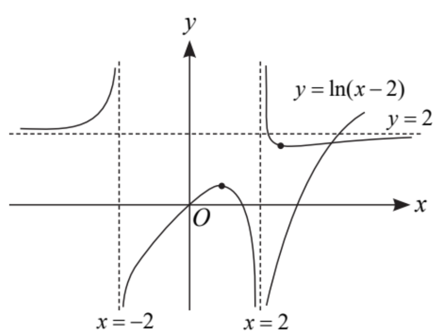


(d) Given $e^y = x - r$

$y = \ln(x - r)$

For $e^y = x - r$ to cut $y = \frac{2x^2 - 3x}{x^2 - 4}$ at one point, the equation $e^y = x - r$ must lie after the vertical asymptote $x = 2$. (See diagram on the right)

$\therefore\ r \geq 2$



(e) $y = \dfrac{2x^2 - 3x}{x^2 - 4}$ ⊲ By performing long division

$y = 2 + \dfrac{8 - 3x}{x^2 - 4}$

$\downarrow$ Replace x by $x+1$

$y = 2 + \dfrac{8 - 3(x+1)}{(x+1)^2 - 4}$

$\downarrow$ Replace x by $-x$

$y = 2 + \dfrac{8 - 3(-x+1)}{(-x+1)^2 - 4}$

$y = 2 + \dfrac{8 - 3(1-x)}{(1-x)^2 - 4}$

Description of sequence of transformations

Translation in negative x direction by 1 unit.

Reflection about the y - axis.

50
Solution

(a) $4x^2+y^2+8mx-4y+4=0$

By completing the square for x terms and y terms,

$$4(x^2+2mx)+y^2-4y+4=0$$

$$4(x+m)^2-4m^2+(y-2)^2=0$$

$$4(x+m)^2+(y-2)^2=4m^2$$

$$\left(\frac{x+m}{m}\right)^2+\left(\frac{y-2}{2m}\right)^2=1$$

$\therefore\ a=m,\ b=m,\ c=-2,\ d=2m$

Method 1

$$x^2+\left(y-\frac{1}{m}\right)^2=1$$

$\downarrow$ Replace x with $\frac{x}{m}$ (1)

$$\left(\frac{x}{m}\right)^2+\left(y-\frac{1}{m}\right)^2=1$$

$\downarrow$ Replace x with $x+m$ (2)

$$\left(\frac{x+m}{m}\right)^2+\left(y-\frac{1}{m}\right)^2=1$$

$\downarrow$ Replace y with $\frac{y}{2m}$ (3)

$$\left(\frac{x+m}{m}\right)^2+\left(\frac{y}{2m}-\frac{1}{m}\right)^2=1$$

$$\left(\frac{x+m}{m}\right)^2+\left(\frac{y-2}{2m}\right)^2=1$$

Description of sequence of transformations

(1) Scaling with scale factor m parallel to the x-axis, followed by

(2) Translation of m units in the negative x-direction, followed by

(3) Scaling with scale factor $2m$ parallel to the y-axis.

Method 2

$$x^2+\left(y-\frac{1}{m}\right)^2=1$$

$\downarrow$ Replace x with $x+1$ (1)

$$(x+1)^2+\left(y-\frac{1}{m}\right)^2=1$$

$\downarrow$ Replace x with $\frac{x}{m}$ (2)

$$\left(\frac{x}{m}+1\right)^2+\left(y-\frac{1}{m}\right)^2=1$$

$\downarrow$ Replace y with $\frac{y}{2m}$ (3)

$$\left(\frac{x}{m}+1\right)^2+\left(\frac{y}{2m}-\frac{1}{m}\right)^2=1$$

$$\left(\frac{x+m}{m}\right)^2+\left(\frac{y-2}{2m}\right)^2=1$$

Description of sequence of transformations

(1) Translation of 1 units in the negative x-direction, followed by

(2) Scaling with scale factor m parallel to the x-axis, followed by

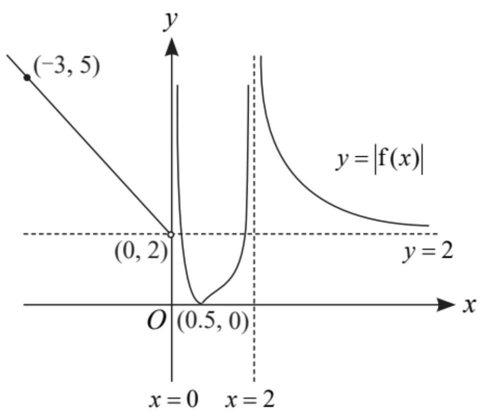
(3) Scaling with scale factor $2m$ parallel to the y-axis.

Exercise 3

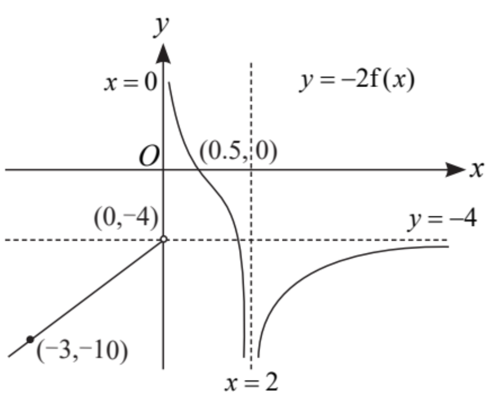
H Exam Style Questions

51
Solution

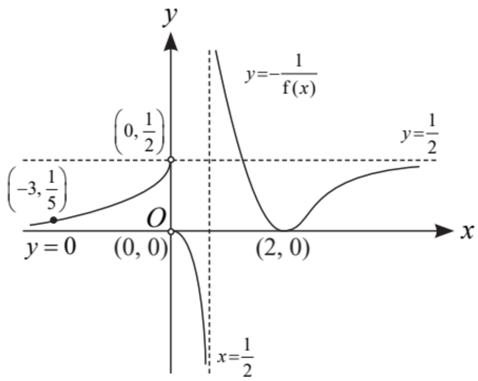
(a) The graph of $y = |f(x)|$



(b) The graph of $y = -2f(x)$

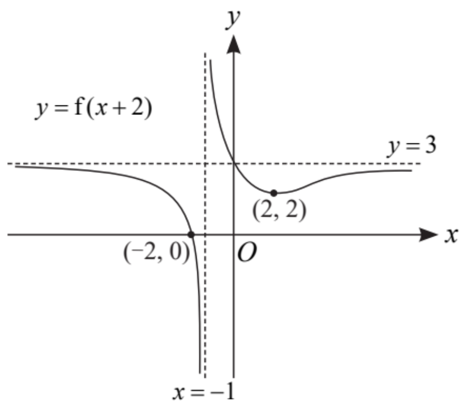


(c) The graph of $y = \dfrac{1}{\mathrm{f}(x)}$

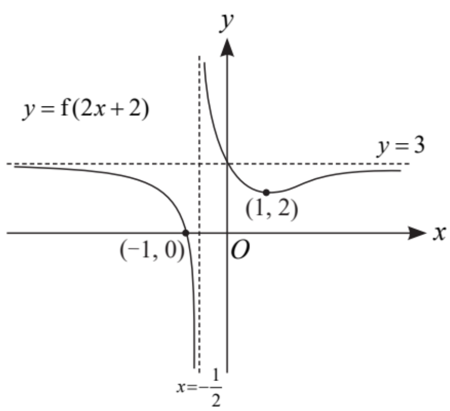


52
Solution

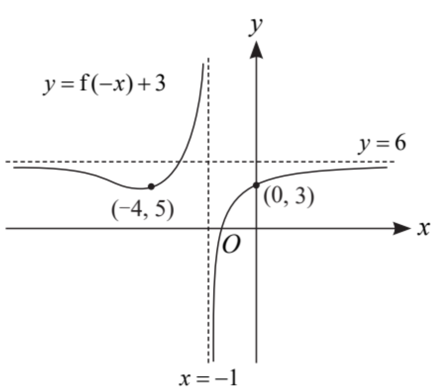
(a) The graph of $y = f(x+2)$



(b) The graph of $y = f(2x+2)$

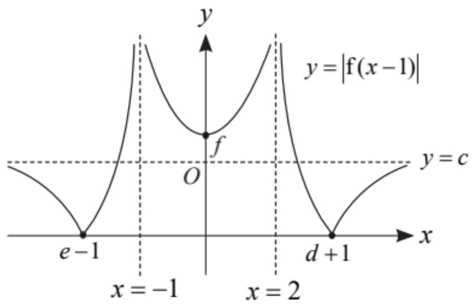


(c) The graph of $y = f(-x)+3$

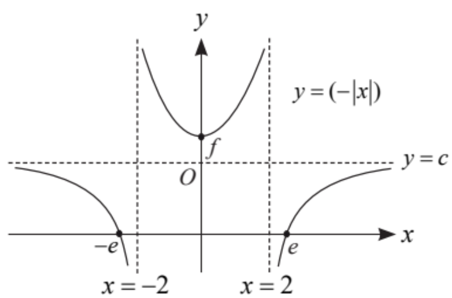


53
Solution

(a) The graph of $y = |\,f(x-1)\,|$

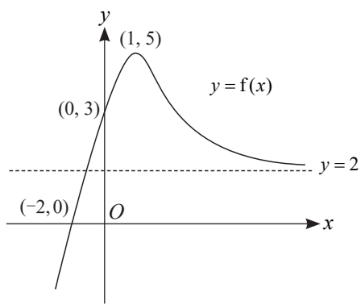


(b) The graph of $y = f(-|x|)$



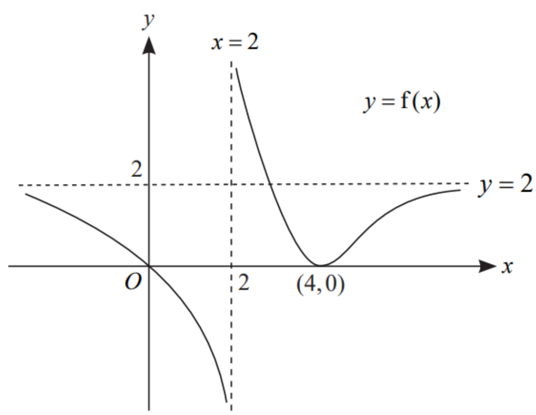
54
Solution

The graph of $y = \mathrm{f}(x)$

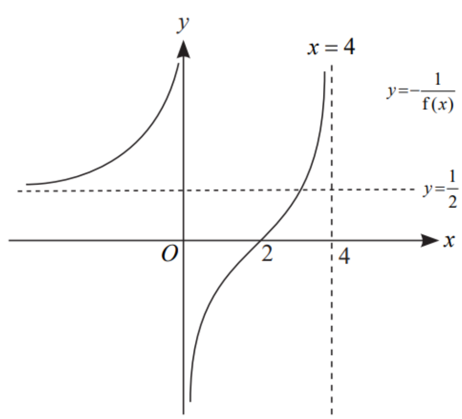


55
Solution

(a) The graph of $y = \mathrm{f}(x)$

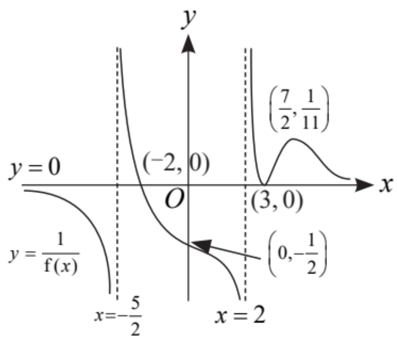


(b) The graph of $y = \dfrac{1}{\mathrm{f}(x)}$

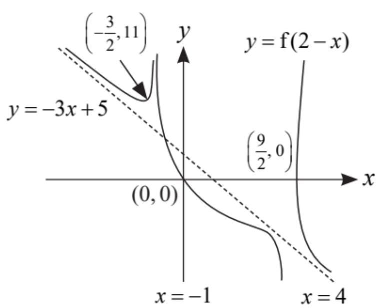


56
Solution

(a)(i) The graph of $y = \dfrac{1}{\mathrm{f}(x)}$



(ii) The graph of $y = \mathrm{f}(2 - x)$



Learning point:

A curve undergoes the following transformations:

$y = \mathrm{f}(x) \longrightarrow y = \mathrm{f}\left(x+2\right) \longrightarrow y = \mathrm{f}\left(-x+2\right)$

To obtain the new oblique asymptote, replace x by $-x+2$ in $y = 3x+1$

i.e. $\quad y = 3(-x+2)+1$

$\quad\quad y = 3x + 5$

(b) $y = \dfrac{1}{x^2 + 4x + 3}$ ◁ complete the square

$= \dfrac{1}{(x+2)^2 - 1}$

$y = \dfrac{1}{(x+2)^2 - 1}$

$\downarrow$ Replace x with $x - 2$

$y = \dfrac{1}{(x-2+2)^2 - 1}$

$y = \dfrac{1}{x^2 - 1}$

$\downarrow$ Replace y with $-y$

$-y = \dfrac{1}{x^2 - 1}$

$y = -\dfrac{1}{x^2 - 1}$

$\downarrow$ Replace y with $y - 3$

$y - 3 = -\dfrac{1}{x^2 - 1}$

$y = 3 - \dfrac{1}{x^2 - 1}$

$y = \dfrac{3x^2 - 4}{x^2 - 1}$

Description of sequence of transformations

Translation of 2 units in positive x - direction, followed by
Reflection in the x - axis, followed by
Translation of 3 units in positive y - direction.

57
Solution

(a) $y = x|-x-a|$

$\downarrow$ (Replace x with $-x$)

$= -x|x+a|$

$\downarrow$ (Replace y with $y-2$)

$y-2 = 2-x|x+a|$

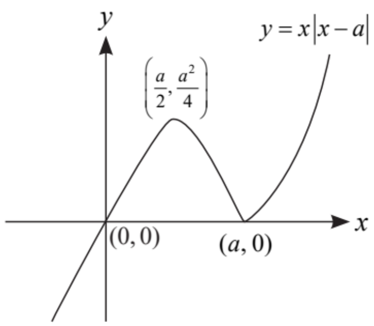
$y = 2-x|x+a|$

Description of sequence of transformations

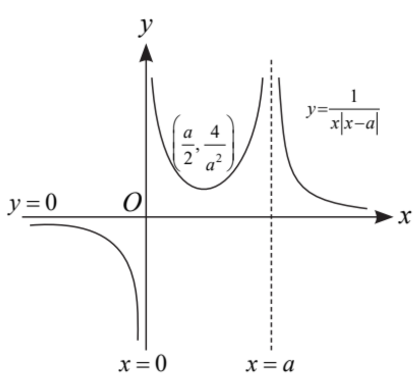
Reflection in the y-axis

Translation of 2 units in the direction of y-axis

(b) The graph of $y = x|x-a|$, $a>0$.



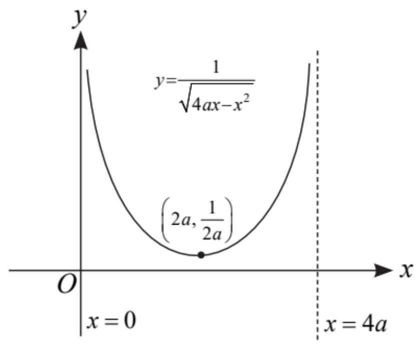
(c) The graph of $y = \dfrac{1}{x|x-a|}$, $a>0$.



(d) The range of values of k in which $\dfrac{1}{x|x-a|} = k$ has exactly one solution is $0<k<\dfrac{4}{a^2}$ or $k<0$.

58
Solution

(a)(i) The graph of $y = \dfrac{1}{\sqrt{4ax - x^2}}$, where $a > 0$.



Learning point:

For $y = \dfrac{1}{\sqrt{4ax - x^2}}$ to exist, $4ax - x^2 > 0$

$\therefore$ vertical asymptotes of the curve occurs at $4ax - x^2 = 0$.

So, $4ax - x^2 = 0$

$x(4a - x) = 0$

$x = 0$ or $x = 4a$

Vertical asymptotes are $x = 0$ or $x = 4a$.

(ii) Using completing the square

Given $4ax - x^2$

$= -(x^2 - 4ax)$

$= -\left[(x - 2a)^2 - 4a^2\right]$

$= 4a^2 - (x - 2a)^2$

The denominator has a maximum value of $\sqrt{4a^2 - (x - 2a)^2}$ occurs at $2a$, where $a > 0$ when $x = 2a$.

Thus the smallest possible value of y is $\dfrac{1}{2a}$.

Alternative Method:

The denominator $4ax - x^2 = x(4a - x)$ is quadratic and is symmetrical in the line $x = 2a$.

And so the largest value of $\sqrt{x(4a - x)}$ occurs at $x = 2a$.

$\therefore$ the largest value is $\sqrt{2a(4a - 2a)} = 2a$, where $a > 0$.

Hence for $y = \dfrac{1}{\sqrt{x(4a - x)}}$, the smallest value y occurs when $x = 2a$.

$\therefore$ the smallest value y is $\dfrac{1}{2a}$.

(iii) Given $y=\dfrac{1}{\sqrt{4ax-x^2}}$

By completing the square,

$$y=\frac{1}{\sqrt{-[(x-2a)^2-4a^2]}}$$

$$y=\frac{1}{\sqrt{4a^2-(x-2a)^2}}$$

$$\downarrow \quad \text{(Replace } x \text{ with } x+2a\text{)}$$

$$y=\frac{1}{\sqrt{4a^2-x^2}}$$

Description of sequence of transformations

The transformation that maps the graph of C onto the graph of $y=\dfrac{1}{\sqrt{4a^2-x^2}}$ is a translation of $2a$ units in the negative x-direction.

(b)

$$y=\ln\left(1-\frac{x}{2}\right)$$

$$\downarrow \quad \text{(Replace } x \text{ with } 2x\text{)}$$

$$y=\ln(1-x)$$

$$\downarrow \quad \text{(Replace } y \text{ with } -y\text{)}$$

$$-y=\ln(1-x)$$

$$y=-\ln(1-x)$$

$$=\ln\left(\frac{1}{1-x}\right)$$

$$\downarrow \quad \text{(Replace } y \text{ with } y-\ln 2\text{)}$$

$$y-\ln 2=\ln\left(\frac{1}{1-x}\right)$$

$$y=\ln\left(\frac{1}{1-x}\right)+\ln 2$$

$$=\ln\left(\frac{2}{1-x}\right)$$

Description of sequence of transformations

Scale the graph of $y=\ln\left(1-\dfrac{x}{2}\right)$ parallel to the x-axis by factor $\dfrac{1}{2}$, followed by

a reflection about the x-axis, followed by

a translation of $\ln 2$ units in the positive y-direction.

59
Solution

(a) $y = x^3 - 4x + 1.$

$\downarrow$ (x replaced by $\frac{1}{4}x$) Transformation C

$$y = \left(\frac{1}{4}x\right)^3 - 4\left(\frac{1}{4}x\right) + 1$$

$$= \frac{1}{64}x^3 - x + 1$$

$\downarrow$ (x replaced by $-x$) Transformation B

$$y = \frac{1}{64}(-x)^3 - (-x) + 1$$

$$y = -\frac{1}{64}x^3 + x + 1$$

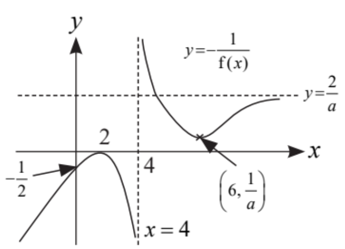
$\downarrow$ (y replaced by $y-1$) Transformation A

$$y - 1 = -\frac{1}{64}x^3 + x + 1$$

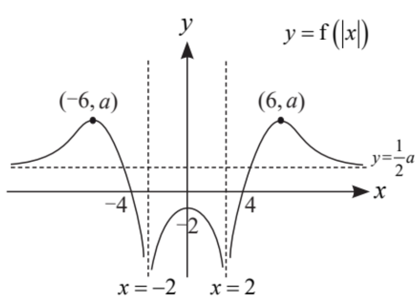
$$y = -\frac{1}{64}x^3 + x + 2$$

$\therefore$ the equation of C_2 is $y = -\frac{1}{64}x^3 + x + 2$

(b)(i) The graph of $y = \frac{1}{\mathrm{f}(x)}$



(ii) The graph of $y = \mathrm{f}(|x|)$



60
Solution

$$y = \frac{3}{x^2 + 2}$$

$\downarrow$ (replace x by $x+1$)

$$y = \frac{3}{(x+1)^2 + 2}$$

$\downarrow$ (replace x by $3x$)

$$y = \frac{3}{(3x+1)^2 + 2}$$

$\downarrow$ (replace x by $-x$)

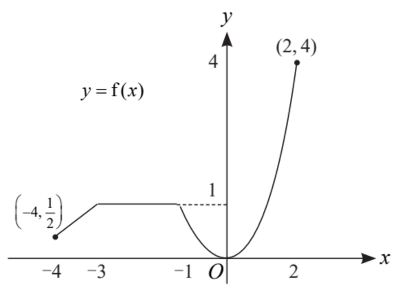
$$= \frac{3}{(-3x+1)^2 + 2}$$

$$= \frac{3}{9x^2 - 6x + 3}$$

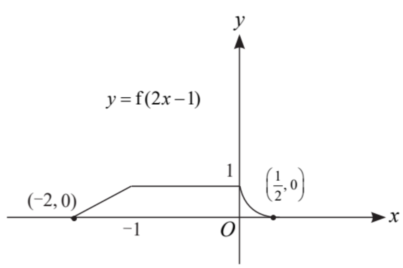
The equation of the resulting curve is $y = \dfrac{1}{3x^2 - 2x + 1}$.

61
Solution

(a) The graph of $y = \mathrm{f}(x)$ for $-4 \le x < 2$.



(b) The graph of $y = \mathrm{f}(2x-1)$, for $-2 \le x \le \dfrac{1}{2}$



62
Solution

(a)

$(y-a)^2 = ax$

$\downarrow$ (Replace x by $x-2$) Transformation A

$(y-a)^2 = a(x-2)$

$\downarrow$ (Replace y by $3y$) Transformation B

$(3y-a)^2 = a(x-2)$

The resulting curve is $(3y-a)^2 = a(x-2)$.

The resulting curve passes through point $\left(2, \frac{4}{3}\right)$.

Substitute $x = 2$ and $y = \frac{4}{3}$ into $(3y-a)^2 = a(x-2)$.

$\therefore \quad (4-a)^2 = a(2-2)$

$(4-a)^2 = 0$

$a = 4$ (Shown)

(b) Substitute $a = 4$ into $(y-a)^2 = ax$.

$\therefore \ (y-4)^2 = 4x$ .... (1)

$y = kx$ .... (2)

Substitute (1) into (2) :

$(kx-4)^2 = 4x$

$k^2x^2 - 8kx + 16 = 4x$

$k^2x^2 + (-8k-4)x + 16 = 0$

For the line not to meet the parabola, Discriminant < 0.

$(-8k-4)^2 - 4k^2(16) < 0$

$64k^2 + 64k + 16k - 64k^2 < 0$

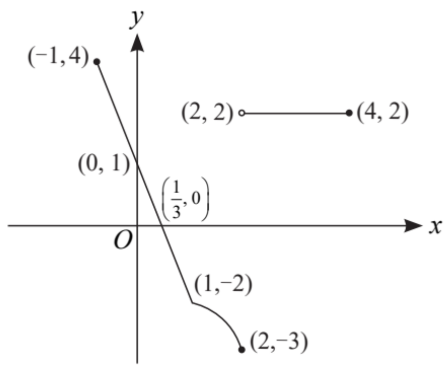
$64k + 16 < 0$

$k < -\frac{1}{4}$

The range of values of k is $k < -\frac{1}{4}$.

63
Solution

(a) The graph of $y = \mathrm{f}(x)$ for $-1 \le x \le 4$



From the graph above, the range of $\mathrm{f} = [-3, 4]$.

(b) $y = \mathrm{e}^{x-2} - x$

$\downarrow C'$: Replace x by $-x$ (Reflect about the y-axis)

$y = \mathrm{e}^{-x-2} - x$

$y = x + \mathrm{e}^{-x-2}$

$\downarrow B'$: Replace x by $5x$ (Scale parallel to x-axis by scale factor of $\frac{1}{5}$)

$y = 5x + \mathrm{e}^{-5x-2}$

$\downarrow A'$: Replace x by $x+1$ (Translate 1 unit in the negative x-direction)

$y = 5(x+1) + \mathrm{e}^{-5(x+1)-2}$

The equation of the original curve $y = 5x + 5 + \mathrm{e}^{-5x-7}$.